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Instabilities in fluid mechanics and convex integration

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Presentación

En esta memoria analizamos varios problemas relacionados con interfases inestables en mecánica de fluidos: el problema del *vortex sheet* y el problema de Muskat totalmente inestable y parcialmente inestable. Además, presentamos un principio de homotopía cuantitativo con aplicaciones a ecuaciones de evolución en el marco de la técnica que emplearemos, el método de integración convexa.

En el capítulo 1 introducimos las ecuaciones en derivadas parciales que estudiaremos, los problemas que han motivado este trabajo, los resultados obtenidos y, por último, el método de integración convexa y el principio de homotopía.

El capítulo 2 está dedicado a reunir algunos aspectos comunes de los problemas que analizamos. Esto nos permite introducir la "zona de turbulencia" donde el fluido se comporta de manera irregular. Esta zona comienza a crecer alrededor de la región inestable de la interfase. Además, presentamos algunos lemas relacionados con los operadores de Birkhoff-Rott y Muskat asociados, los cuales serán cruciales en las pruebas de los resultados de los capítulos 4-6. Estos lemas pueden encontrarse en [103], trabajo conjunto con László Székelyhidi, y en [27], trabajo conjunto con Ángel Castro y Daniel Faraco.

En el capítulo 3 presentamos una versión cuantitativa del principio de homotopía para cierta clase de ecuaciones de evolución. En las versiones previas ([52]) se recuperan soluciones débiles a partir de una "subsolución" a través de un esquema de integración convexa. Esta versión cuantitativa recupera además información microscópica y macroscópica. Por un lado, muestra que el fluido puede comportarse de manera muy irregular dentro de la zona de turbulencia. Por otro lado, mide la proximidad, en términos de cantidades débilmente*-continuas, entre las soluciones débiles obtenidas y la subsolución. Esto permite seleccionar aquellas soluciones que retienen más información de la subsolución y, por tanto, enfatizar el papel de la subsolución como candidata a solución macroscópica. Estos resultados se basan en [26], trabajo conjunto con Ángel Castro y Daniel Faraco.

En el capítulo 4 estudiamos el problema del *vortex sheet* para la ecuación de Euler incompresible. Construimos infinitas soluciones admisibles comenzando con vorticidades sin signo fijo y concentradas en interfases no analíticas. La existencia de soluciones débiles para datos tipo *vortex sheet* con signo mixto era un problema abierto desde el trabajo de Delort ([56]) y solo se conocía para datos iniciales particulares ([94]). Estas soluciones débiles son suaves fuera de una zona de turbulencia que crece linealmente en tiempo alrededor del *vortex sheet*. Además, este enfoque muestra cómo el crecimiento de la zona de turbulencia es controlado por la desigualdad de energía local, y mide la tasa máxima de disipación inicial en términos de la densidad de vorticidad. Estos resultados aparecen en [103], trabajo conjunto con László Székelyhidi.

En el capítulo 5 probamos un principio de homotopía para la ecuación de los medios porosos incompresible (IPM) con salto de densidad y viscosidad. Como primer ejemplo, obtenemos soluciones no triviales con soporte compacto en tiempo (véase [39] para el caso de viscosidad constante). En segundo lugar, construimos soluciones mezcla para el problema de Muskat inestable con interfase plana. Como corolario comprobamos que la conexión, establecida por Székelyhidi para el caso de viscosidad constante ([126]), entre la subsolución y la solución relajada Lagrangiana de Otto ([116]), se satisface también en el caso de salto de viscosidad. En este caso mostramos cómo una singularidad de pellizco en la relajación impide que los fluidos se mezclen dondequiera que no haya ni inestabilidad de Rayleigh-Taylor ni vorticidad en la interfase.

En el capítulo 6 construimos soluciones mezcla para IPM comenzando con datos tipo Muskat en el régimen parcialmente inestable. En particular, consideramos interfases burbuja e interfases giradas con regularidad Sobolev. Como corolario, probamos la continuación de la evolución de IPM tras la ruptura de la condición de Rayleigh-Taylor y de la analiticidad exhibidas en [24, 23]. En cada instante de tiempo, el espacio se divide en tres dominios que evolucionan: dos zonas no mezcladas y una zona de mezcla localizada en un entorno de la región inestable. De esta manera, mostramos la compatibilidad entre el problema de Muskat clásico y el método de integración convexa. Estos resultados aparecen en [27], trabajo conjunto con Ángel Castro y Daniel Faraco.

Conclusiones

La versión del método de integración convexa introducida en Hidrodinámica por De Lellis y Székelyhidi ha demostrado ser una herramienta muy poderosa en la última década. En esta memoria aplicamos este método para resolver varios problemas relacionados con interfases inestables, creando una "zona de turbulencia" alrededor de la región inestable de la interfase.

En primer lugar, conjuntamente con Ángel Castro y Daniel Faraco ([26]) presentamos una versión cuantitativa del principio de homotopía para una clase de ecuaciones de evolución. Esta versión permite recuperar información microscópica y macroscópica de las soluciones a partir de la "subsolución" dentro de la zona de turbulencia. A pesar de la falta de unicidad y regularidad de las soluciones dentro de la zona de turbulencia, mostramos que estas soluciones son esencialmente indistinguibles de la subsolución a nivel macroscópico.

En segundo lugar, estudiamos el problema del *vortex sheet* para la ecuación de Euler incompresible. Construimos, conjuntamente con László Székelyhidi ([103]) infinitas soluciones admisibles comenzando con vorticidades sin signo fijo y concentradas en interfases no analíticas. Además, mostramos la relación entre la disipación de energía y el crecimiento de la zona de turbulencia.

En tercer lugar, estudiamos el problema de Muskat para la ecuación de los medios porosos incompresible. Para el caso de distintas densidades y viscosidades, probamos un principio de homotopía ([102]) que permite relacionar la subsolución de la interfase plana con la solución relajada Lagrangiana de Otto. Para el caso de distintas densidades y viscosidad constante, construimos, conjuntamente con Ángel Castro y Daniel Faraco ([27]) soluciones mezcla comenzando en el régimen parcialmente inestable. Para ello, combinamos el análisis parabólico en la región estable con el método de integración convexa localizando la zona de mezcla alrededor de la región inestable.

Quedan todavía muchas preguntas abiertas. Una de ellas es determinar criterios de selección que permitan recuperar una única subsolución física. Otro problema relacionado es ver si estas subsoluciones pueden obtenerse como el caso límite de mecanismos de regularización, tales como la tensión superficial, métodos de vorticidad y viscosidad, u otros. Otra pregunta interesante es entender y alcanzar el umbral de regularidad de las soluciones dentro de la zona de turbulencia. Nos gustaría explorar estas cuestiones en futuros trabajos.

Abstract

In this dissertation we analyze several problems related to unstable interfaces in fluid mechanics: the vortex sheet problem and both the fully unstable and partially unstable Muskat problem. Moreover, we present a quantitative homotopy principle with applications to evolution equations within the framework of the technique we will employed, the convex integration method.

In chapter 1 we introduce the partial differential equations that we will study, the problems that have motivated this dissertation, the results we have obtained in [26, 103, 102, 27] and, finally, the convex integration method and the homotopy principle.

The chapter 2 is devoted to gather some common aspects of the problems we analyze. This allows us to introduce the "turbulence zone" where the fluid behaves wildly. This zone starts to grow around the unstable region of the interface. Moreover, we present some lemmas related to the Birkhoff-Rott and Muskat operators, which will be crucial within the proofs of the results in chapters 4-6. These lemmas can be found in [103], joint work with László Székelyhidi, and in [27], joint work with Ángel Castro and Daniel Faraco.

In chapter 3 we present a quantitative version of the homotopy principle for a class of evolution equations. In the previous versions ([52]) weak solutions are recovered from a "subsolution" through a convex integration scheme. This quantitative version recovers also microscopic and macroscopic information. On the one hand, it shows that the fluid can behave wildly inside the turbulence zone. On the other hand, it measures the proximity, in terms of weak*-continuous quantities, between the weak solutions obtained and the subsolution. This allows to select those which retain more information from the subsolution, thus emphasizing the role of the subsolution as the candidate for the macroscopic solution. These results are based on [26], joint work with Ángel Castro and Daniel Faraco.

In chapter 4 we study the vortex sheet problem for the incompressible Euler equation. We construct infinitely many admissible solutions starting from vorticities without fixed sign and concentrated on non-analytic curves. The existence of weak solutions for vortex sheet data with mixed sign was an open problem from the work of Delort ([56]) and was only known for particular initial data ([94]). These weak solutions are smooth outside a turbulence zone which grows linearly in time around the vortex sheet. Furthermore, this approach shows how the growth of the turbulence zone is controlled by the local energy inequality, and measures the maximal initial dissipation rate in terms of the vortex sheet strength. These results appear in [103], joint work with László Székelyhidi.

In chapter 5 we prove a homotopy principle for the incompressible porous media (IPM) equation with density-viscosity jump. As a first example, non-trivial weak solutions with compact support in time are obtained (see [39] for the case of constant viscosity). Secondly, we construct mixing solutions to the unstable Muskat problem with initial flat interface. As a byproduct, we

check that the connection, established by Székelyhidi for the case of constant viscosity ([126]), between the subsolution and the Lagrangian relaxed solution of Otto ([116]), holds for the case of viscosity jump as well. In this case we show how a pinch singularity in the relaxation prevents the two fluids from mixing wherever there is neither Rayleigh-Taylor instability nor vorticity at the interface.

In chapter 6 we construct mixing solutions to the incompressible porous media equation starting from Muskat type data in the partially unstable regime. In particular, we consider bubble and turned type interfaces with Sobolev regularity. As a by-product, we prove the continuation of the evolution of IPM after the Rayleigh-Taylor and smoothness breakdown exhibited in [24, 23]. At each time slice the space is split into three evolving domains: two non-mixing zones and a mixing zone which is localized in a neighborhood of the unstable region. In this way, we show the compatibility between the classical Muskat problem and the convex integration method. These results appear in [27], joint work with Ángel Castro and Daniel Faraco.

Conclusions

The version of the convex integration method introduced in Hydrodynamics by De Lellis and Székelyhidi has been proved to be a very powerful tool in the last decade. In this dissertation we apply this method to solve several problems related to unstable interfaces by creating a "turbulence zone" around the unstable region of the interface.

Firstly, jointly with Ángel Castro and Daniel Faraco ([26]) we present a quantitative version of the homotopy principle for a class of evolution equations. This version allows us to recover microscopic and macroscopic information of the solutions from the "subsolution" inside the turbulence zone. In spite of the lack of uniqueness and regularity of the solutions inside the turbulence zone, we show that these solutions are essentially indistinguishable from the subsolution at a macroscopic level.

Secondly, we study the vortex sheet problem for the incompressible Euler equation. We construct, jointly with László Székelyhidi ([103]) infinitely many admissible solutions starting from vorticities without fixed sign and concentrated on non-analytic curves. Moreover, we show the relation between the dissipation of energy and the growth of the turbulence zone.

Thirdly, we study the Muskat problem for the incompressible porous media equation. For the case of different densities and viscosities, we prove an h-principle ([102]) which allows us to relate the subsolution of the planar interface with the Lagrangian relaxed solution of Otto. For the case of different densities and constant viscosity, we construct, jointly with Ángel Castro and Daniel Faraco ([27]) mixing solutions starting from the partially unstable regime. To do this, we combine the parabolic analysis in the stable region with the convex integration method by localizing the mixing zone around the unstable region.

There are still many open questions. One of them is to determine selection criteria which allow us to recover a unique physical subsolution. Other related problem is to show if these subsolutions can be obtained as the limiting case of regularizing mechanisms as the surface tension, vortex and viscosity methods, or others. Another interesting question is to understand and achieve the regularity threshold of the solutions inside the turbulence zone. We would like to explore these questions in future works.

Chapter 1

Introduction and main results

The aim of this chapter is to introduce the PDEs we will study, the problems that have motivated this dissertation, the results we have obtained in [26, 103, 102, 27], and the technique employed.

1.1 Brief introduction to Fluid Mechanics

A fluid is a substance that deforms under an applied shear stress or external force, commonly liquids and gases. Given that the vast majority of the observable mass in the universe exists in a fluid state, and that life as we know is not possible without rivers, oceans, the atmosphere, biofluids (e.g. the blood), etc. fluid mechanics has unquestioned scientific and practical relevance ([89, sec. 1.1]).

Undoubtedly, water is one of the most important as it is an essential element for life and since it covers the Earth. Note that, even in a glass of water, there is an astronomic number of particles in constant motion undergoing collisions with each other. Thus, it becomes more practical to ignore the discrete molecular structure and replace it with a continuous distribution, called a continuum ([89, sec. 1.4]).

Under this continuum hypothesis, the movement of a smooth flow filling a fixed domain $\mathscr{D} \subset \mathbb{R}^n$ $(n = 2, 3)$ can be described by the **particle trajectory map** of the fluid

$$X : \mathbb{R}_+ \times \mathscr{D} \to \mathscr{D},$$

where $X(t, x)$ represents the position at time t of any fluid element $x = (x_1, \ldots, x_n) \in \mathscr{D}$. Equivalently, it can be described by the **velocity field** of the fluid

$$v : \mathbb{R}_+ \times \mathscr{D} \to \mathbb{R}^n.$$

The particle trajectory map and the velocity field are related through the ODE

$$\left\{ \begin{array}{l} \frac{\mathrm{d}}{\mathrm{d}t} X(t,x) = v(t, X(t,x)), \\ X(0,x) = x. \end{array} \right. \tag{1.1}$$

Then, a physical property ϕ of the fluid evolves from its initial state ϕ° in such a way that

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t} \int_{X(t,\Omega)} \phi(t,x)\,\mathrm{d}x}_{\text{rate of change of } \phi} = \underbrace{\int_{\partial X(t,\Omega)} s \cdot \nu(t,\alpha)\,\mathrm{d}\alpha + \int_{X(t,\Omega)} b(t,x)\,\mathrm{d}x}_{\text{surface } (s) \text{ and body } (b) \text{ sinks and sources}}, \tag{1.2}$$

is satisfied on every control volume $\Omega \subset \mathscr{D}$, where ν is the normal vector to the boundary $\partial X(t,\Omega)$ pointing outward.

On the one hand, Gauss divergence theorem says that

$$\int_{\partial X(t,\Omega)} s \cdot \nu(t,\alpha)\,\mathrm{d}\alpha = \int_{X(t,\Omega)} \mathrm{div} s(t,x)\,\mathrm{d}x,$$

where $\mathrm{div} s := \nabla \cdot s$ is the divergence operator (for the x variable). On the other hand, Reynolds transport theorem[1] says that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{X(t,\Omega)} \phi(t,x)\,\mathrm{d}x = \int_{X(t,\Omega)} \left(\partial_t\phi + \mathrm{div}(\phi v)\right)(t,x)\,\mathrm{d}x.$$

Since (1.2) must be satisfied on every control volume Ω, this is equivalent to the PDE

$$\partial_t\phi + \mathrm{div}(\phi v) = \mathrm{div} s + b. \tag{1.3}$$

A scalar property $\theta : \mathbb{R}_+ \times \mathscr{D} \to \mathbb{R}$ which is simply advected by the flow $(s, b = 0)$ obeys

$$\partial_t\theta + \mathrm{div}(\theta v) = 0. \tag{1.4}$$

One of these properties is the **density**, denoted by ρ, for which (1.4) reads as the **conservation of mass** equation. In this dissertation we will focus on **incompressible** fluids, those preserving volume ($\theta = 1$) for which (1.4) reads as the incompressibility condition

$$\mathrm{div} v = 0. \tag{1.5}$$

We note in passing that, for smooth incompressible fluids, the advection equation (1.4) is equivalent to the transport equation

$$\partial_t\theta + v \cdot \nabla\theta = 0.$$

In such a case, the solution is given by $\theta(t, X(t,x)) = \theta^\circ(x)$. However, in the problems we will consider the velocity field v is not regular enough to define the map X through the ODE (1.1), and thus we shall interpret (1.4) in the sense of distributions.

Finally, the velocity field v is determined by Newton's second law, as it states that the **momentum** ρv changes under the influence of forces, or translated into (1.2): b equals external body forces $f : \mathbb{R}_+ \times \mathscr{D} \to \mathbb{R}^n$ (e.g. the gravity) and s equals internal surface forces $\sigma : \mathbb{R}_+ \times \mathscr{D} \to \mathbb{R}^{n\times n}$ (Cauchy's stress tensor). In addition, σ is split into $\sigma = -pI_n + \tau$ where I_n is the identity matrix of size n, $p : \mathbb{R}_+ \times \mathscr{D} \to \mathbb{R}$ is the hydrostatic **pressure** and $\tau : \mathbb{R}_+ \times \mathscr{D} \to \mathbb{R}^{n\times n}$ the shear stress coming from the resistance of the fluid to be deformed. For the so-called Newtonian fluids (e.g. water and air) the shear stress is proportional to the rate of deformation, $\tau = \mu(\nabla v + (\nabla v)^\dagger)$ where μ is the dynamic **viscosity**. In such a case, (1.3) reads as the **conservation of momentum** (vector) equation

$$\partial_t(\rho v) + \mathrm{div}(\rho v \otimes v) = -\nabla p + \mu\Delta v + f, \tag{1.6}$$

where $v \otimes v := vv^\dagger = (v_i v_j)$ and $\Delta v := \nabla \cdot \nabla v$ is the Laplace operator. This system of PDEs is known as the (inhomogeneous) incompressible **Navier-Stokes** equation. Although this is valid for n-dimensional flows, in this dissertation we will focus on the planar case $n = 2$.

We finish mentioning that these equations admit generalizations in other contexts (see e.g. [89]). However, for our purposes this level of generality is adequate.

[1] Alternatively, it can be deduced by changing variables $x = X(t,x^\circ)$ and using the identity $\partial_t \log J(t,x) = (\mathrm{div} v)(t, X(t,x))$ where $J \equiv \det(\nabla X) > 0$ ([96, Prop. 1.2]).

1.1.1 Incompressible Euler equation

In this section we deal with homogeneous fluids, those for which ρ and μ are constant. In this case the conservation of mass is equivalent to the incompressibility condition ($\mathrm{div} v = 0$). We will assume w.l.o.g. that $\rho = 1$. In addition, we consider $\mathscr{D} = \mathbb{R}^2$ and that there are not body forces $f = 0$. This corresponds to the incompressible Navier-Stokes equation

$$\partial_t v + \mathrm{div}(v \otimes v) = -\nabla p + \mu \Delta v, \tag{1.7a}$$

$$\mathrm{div} v = 0. \tag{1.7b}$$

For smooth flows we can multiply the conservation of momentum (1.7a) by the velocity v, which yields the following equation for the **kinetic energy** density $e := \frac{1}{2}|v|^2$

$$\partial_t e + \mathrm{div}((e + p)v) = \mu v \cdot \Delta v. \tag{1.8}$$

If the flow decays fast enough, we can integrate (1.8) to deduce that the global kinetic energy

$$E(t) := \int_{\mathbb{R}^2} e(t, x)\,\mathrm{d}x$$

is dissipated for $\mu > 0$ according to

$$\frac{\mathrm{d}}{\mathrm{d}t} E = -\mu \int_{\mathbb{R}^2} |\nabla v|^2\,\mathrm{d}x.$$

Hence, one would expect E to be conserved in the inviscid limit $\mu \to 0$. In fact, this is the case of smooth flows. However, it is observed that the mean **energy dissipation rate** remains strictly positive and independent of the viscosity as $\mu \to 0$ for suitable averages of turbulent flows (see e.g. [124, 118]). This is known as the zeroth law of turbulence, or anomalous dissipation of energy ([71, chapter 5]) which motivates to analyze directly the case of ideal fluids $\mu = 0$. This corresponds to the incompressible Euler equation ([62])

$$\partial_t v + \mathrm{div}(v \otimes v) = -\nabla p, \tag{1.9a}$$

$$\mathrm{div} v = 0. \tag{1.9b}$$

Following the work of Duchon and Robert ([60]), we introduce the dissipation measure associated to a weak solution of the incompressible Euler equation, the distribution D given by

$$\partial_t e + \mathrm{div}((e + p)v) =: -D. \tag{1.10}$$

As we mentioned, for smooth flows it holds that $D = 0$, while for turbulent flows it is expected that $D > 0$ ([60]). If the flow decays fast enough, we can test (1.10) to deduce that

$$E = E(0) - \langle D, \mathbb{1} \rangle.$$

In the celebrated paper [114] Onsager conjectured the Hölder regularity threshold, this is $1/3$, beyond which E is conserved or may decrease (in the 3D periodic box). The regular part of Onsager's conjecture was proved by Constantin, E and Titi, and by Eyink in [34, 64] respectively. The dissipative part was proved by Isett, and by Buckmaster, De Lellis, Székelyhidi and Vicol in [81, 13] respectively through the convex integration method.

In this regard, we would like to know the initial data for which there are weak solutions dissipating the global kinetic energy. In [127] Székelyhidi constructed such dissipative solutions starting from the planar vortex sheet. This result motivated to extend the construction for more general vortex sheet initial data.

The vortex sheet problem

The vortex sheet dynamics serve as a simplified model of many physical phenomena observed in the atmosphere and oceans related to turbulence, such as mixing layers, jets and wakes (see [96, sec. 9] and [132]). By neglecting the effects of surface tension and viscosity, this predicts the evolution of two incompressible and irrotational fluids (e.g. two masses of water) when they come into contact with different motions ([10]). This discontinuity induces vorticity $\omega^\circ := \nabla^\perp \cdot v^\circ = \partial_1 v_2^\circ - \partial_2 v_1^\circ$ at the interface z° with some vorticity strength ϖ° ($\omega^\circ = \varpi^\circ \delta_{z^\circ}$).

The initial velocity v°, which is recovered from ω° through the Biot-Savart law, is smooth outside z° but has tangential discontinuities along it. Under the assumption that the vorticity remains concentrated on a movable interface z for later times, the incompressible Euler equation turns into a Cauchy problem for z in terms of the Birkhoff-Rott operator, the so-called vortex sheet problem. Roughly speaking, this Cauchy problem for the interface is ill-posed unless the initial data is real-analytic or well prepared (see e.g. [135]). By desingularizing the Birkhoff-Rott operator, it can be observed that the interface tends to roll-up into spiral vortices ([87]). This phenomenon is known as the **Kelvin-Helmholtz** instability.

In spite of the fact that the interface evolution is ill-posed, there is still hope to solve the incompressible Euler equation in a different manner. In the celebrated paper [56], Delort constructed global weak solutions to the incompressible Euler equation starting from vorticities in the class $D^+ := \mathcal{M}^+ \cap H^{-1}$, where $\mathcal{M}^+$ denotes the space of positive Radon measures. Of course this applies also for negative vorticities. Delort's class particularly includes the case of vortex sheets whose vorticity strength has a fixed sign. The case of mixed sign vortex sheets has its own interest, both for its practical applications in aerodynamics and for the complex structures created by the intertwining between regions of positive and negative vorticity ([86]). Despite this, Delort's result has been only extended to $L^p + D^+$ ([56, 121, 133]) and to the case of vortex sheets changing sign with reflection symmetry by Lopes, Nussenzveig and Xin ([94]).

Observe that, for those vortex sheets in the Delort class for which the interface evolution is ill-posed, Delort's result provides Euler flows whose vorticity cannot be accumulated in a regular curve. This suggests to consider a "turbulence zone" Ω_{tur}, a region where the vorticity is supported and the fluid may behave wildly. In [127] Székelyhidi constructed this turbulence zone for the planar vortex sheet in the 2D periodic box. In this case we have $z^\circ(\alpha) = (\alpha, 0)$ and $\varpi^\circ(\alpha) = 2$, and thus v° corresponds to the shear flow

$$v^\circ(x) = \begin{cases} (+1, 0), & x_2 > 0, \\ (-1, 0), & x_2 < 0. \end{cases}$$

For later times the turbulence zone grows linearly in time around z° with constant growth rate $0 < c < 1$. Remarkably, the global dissipation rate is related to c via

$$\frac{\mathrm{d}}{\mathrm{d}t} E = -\tfrac{2}{3} c(1-c).$$

Therefore, the global dissipation rate is maximized $\frac{\mathrm{d}}{\mathrm{d}t}E = -\frac{1}{6}$ for $c = \frac{1}{2}$, thus providing a selection criterion for the growth rate of the turbulence zone.

In [103] we construct, jointly with László Székelyhidi, admissible weak solutions to the incompressible Euler equation starting from vorticities without fixed sign and concentrated on non-analytic chord-arc curves (cf. Def. 2.1.1). We remark that the global kinetic energy E is not well defined in $\mathbb{R}^2$ unless ϖ° has zero mean. In general, we say that a weak solution, whose dissipation D is compactly supported, is admissible if $\langle D, \mathbb{1}\rangle \geq 0$. For completeness, we will check that this condition guarantees weak-strong uniqueness. These dissipative solutions are recovered by the convex integration method applied in Ω_{tur} to a so-called "subsolution", which is intended to be a kind of coarse-grained solution to the incompressible Euler equation.

With these preparations, our main result is as follows.

Theorem 1.1.1. *Let $z^\circ \in C^{5,\delta}(\mathbb{T};\mathbb{R}^2)$ be a closed chord-arc curve and $\varpi^\circ \in C^{4,\delta}(\mathbb{T};\mathbb{R})$ not identically zero, for some $\delta > 0$, and let $N \in \mathbb{N}$. There exist infinitely many admissible weak solutions to the incompressible Euler equation starting from the vortex sheet initial datum $\omega^\circ = \varpi^\circ\delta_{z^\circ}$. Moreover, the rate of global dissipation and growth $c(\alpha)$ of the turbulence zone are related via*

$$\left\langle \frac{D(t_2)-D(t_1)}{t_2-t_1}, \mathbb{1} \right\rangle = \tfrac{1}{3}\tfrac{2N+1}{2N-1}\int_{\mathbb{T}} c|\varpi^\circ|\left(\tfrac{2N-1}{4N}|\varpi^\circ| - c\right)\mathrm{d}\alpha + O(t_2), \tag{1.11}$$

for all $0 \leq t_1 < t_2 \leq T$, where T depends on $\|\varpi^\circ\|_{C^{4,\delta}}$, $\|z^\circ\|_{C^{5,\delta}}$ and the chord-arc condition.

For zero-mean ϖ°'s, (1.11) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}E = -\tfrac{1}{3}\tfrac{2N+1}{2N-1}\int_{\mathbb{T}} c|\varpi^\circ|\left(\tfrac{2N-1}{4N}|\varpi^\circ| - c\right)\mathrm{d}\alpha + O(t).$$

As in [127], the global dissipation rate is maximized at $t=0$ as $c(\alpha) \to \frac{1}{4}|\varpi^\circ(\alpha)|$ and $N \to \infty$, and equals

$$\frac{\mathrm{d}}{\mathrm{d}t}E|_{t=0} = -\tfrac{1}{48}\|\varpi^\circ\|_{L^3}^3.$$

Finally, in order to prevent local creation of kinetic energy along z°, we will test D with a larger class of ψ's, instead of just $\psi = \mathbb{1}$. Thus, we will obtain a local version of Theorem 1.11, which is presented in chapter 4 (Theorem 4.1.3).

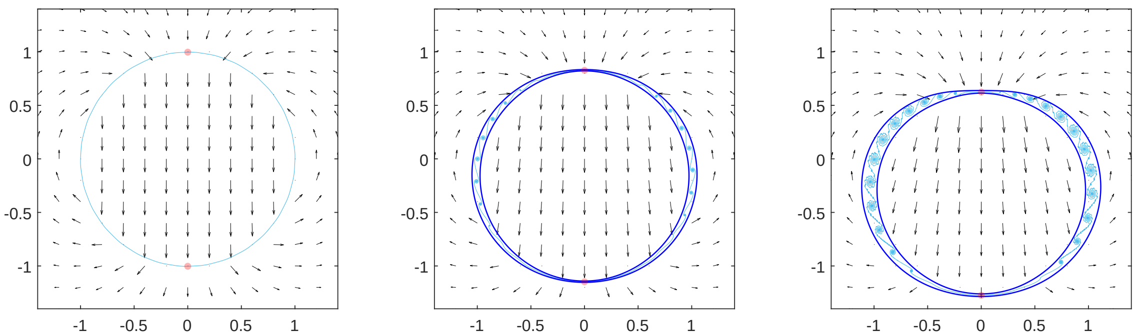


Figure 1.1: Left: The divergence-free velocity field v° with vorticity strength $\varpi^\circ(\alpha) = \frac{1}{4}\cos(\alpha)$ along the interface $z^\circ(\alpha) = e^{i\alpha}$. Center-Right: At some $t > 0$, the velocity field $v(t)$ outside $\Omega_{\mathrm{tur}}(t)$, the boundary of the turbulence zone $z_\pm(t) = z(t) \pm tc\partial_\alpha z^{\circ\perp}$ (dark blue) for some $z(t,\alpha)$ and $c(\alpha) \propto |\varpi^\circ(\alpha)|$. Inside $\Omega_{\mathrm{tur}}(t)$ we plot the Kelvin-Helmholtz instability (light blue).

1.1.2 Incompressible Porous Media equation

In this section we deal with fluids moving under the action of gravity[2] $f = -\rho g(0,1)$, through a homogeneous porous medium, $\mathscr{D}_{\mathrm{PM}} = \mathscr{D} \setminus \mathscr{D}_{\mathrm{S}}$ with $\mathscr{D}_{\mathrm{S}}$ representing the solid part, so that the size of the obstacles, say $s > 0$, and the remaining pores are comparable. As the Navier-Stokes equation becomes cumbersome on the complicated domain $\mathscr{D}_{\mathrm{PM}}$, it is more practical to consider an asympthotic model on $\mathscr{D}$ as $s \to 0$. A widely accepted model consists in replacing the conservation of momentum equation (1.6) by **Darcy's law**

$$\frac{\mu}{\kappa} v = -\nabla p - \rho g(0,1), \tag{1.12}$$

where $\kappa > 0$ represents the **permeability** of the porous medium. This law was obtained empirically by Darcy in [49]. Leter, this was also derived from the Navier-Stokes equation using homogenization ([130][3]). As a result, the velocity is not driven by an evolution equation but determined at each time slice by the density, the viscosity and changes in the pressure.

This law is also valid for fluids trapped between two fixed parallel plates with separation $b > 0$, called a Hele-Shaw cell $\mathscr{D}_{\mathrm{HS}} = \mathscr{D} \times (0,b)$. For b small enough the fluid is essentially planar, and (1.6) is simplified by (1.12) on $\mathscr{D}$ with $\kappa = b^2/12$ ([119]). We will assume w.l.o.g. that $\kappa = g = 1$.

In [109] the petroleum engineer Morris Muskat became interested in the case of two fluids with constant densities[4] $\rho_h > \rho_l$ and viscosities μ_h, μ_l, moving through a homogeneous porous medium (e.g. water and oil in sand). In this setting it is convenient to introduce the $\{-1,1\}$-valued variable $\theta(t,x)$ to indicate whether at time $t \in \mathbb{R}_+$ the pores near $x = (x_1, x_2) \in \mathscr{D}$ are filled with **phase** l or h

$$a(t,x) := \frac{a_h + a_l}{2} + \frac{a_h - a_l}{2}\theta(t,x), \qquad a = \rho, \mu. \tag{1.13}$$

In this case, the incompressible porous media (IPM) equation reads as

$$\partial_t \theta + \operatorname{div}(\theta v) = 0, \tag{1.14}$$
$$\operatorname{div} v = 0, \tag{1.15}$$
$$\mu v + \rho(0,1) = -\nabla p, \tag{1.16}$$

with ρ, μ given in terms of θ via (1.13). By (1.14)(1.15), ρ, μ are conserved properties. We remark that p will be recovered through Potential theory and thus we will focus on (θ, v) by taking $\mathrm{rot} := \nabla^{\perp}\cdot$ on Darcy's law (1.16).

Firstly, we extend the result in [39] on the lack of uniqueness to IPM in the case of constant viscosity $\mu_h = \mu_l$, to the case of viscosity jump.

Theorem 1.1.2. *Let $\rho_h > \rho_l$, μ_h, μ_l, $T > 0$ and $\mathscr{D} = \mathbb{R}^2$ or $\mathbb{T}^2$. There exist infinitely many weak solutions $(\theta, v) \in C(\mathbb{R}_+; L^\infty_{w^*}(\mathscr{D}))$ to IPM with $|\theta| = 1$ on $(0,T) \times \mathscr{D}$ and $\theta = 0$ outside.*

[2] $(0,1) \equiv$ unit vertical vector.
[3] A heuristic argument: the acceleration term (l.h.s. of (1.6)) is neglected because of the friction caused by the pores ($v = 0$ on $\partial\mathscr{D}_{\mathrm{PM}}$) and Δv behaves as $-v/\kappa$.
[4] $h \equiv$ heavier, $l \equiv$ lighter.

Thus, IPM admits non-trivial weak solutions with compact support in time. Opposite to these paradoxical examples, we construct admissible solutions to the unstable Muskat problem with initial flat interface $z^\circ(\alpha) = (\alpha, 0)$. This is IPM starting from the unstable planar phase

$$\theta^\circ(x) = \begin{cases} +1, & x_2 > 0, \\ -1, & x_2 < 0. \end{cases} \tag{1.17}$$

Similarly to [126, 22, 26], we show that these weak solutions start to mix inside a "mixing zone" Ω_{mix} which grows linearly in time around z°, and that they look macroscopically almost like the coarse-grained phase, denoted in this dissertation by Θ_A where $A = \frac{\mu_h - \mu_l}{\mu_h + \mu_l}$ (Atwood number), introduced by Otto in [116]. For this reason, we will call them "Θ_A-mixing solutions". This extends the previous result of Székelyhidi for the case of constant viscosity ([126]). In addition, we will estimate the volume proportion of each fluid in every rectangle of the mixing zone.

Theorem 1.1.3. *Let $\rho_h > \rho_l$, μ_h, μ_l and $\mathscr{D} = \mathbb{R}^2$ or $(-1,1)^2$. There exist infinitely many Θ_A-mixing solutions $(\theta, v) \in C(\mathbb{R}_+; L^\infty_{w*}(\mathscr{D}))$ to IPM starting from* (1.17).

These theorems are deduced from a more general h-principle which will be presented in chapter 5 (Theorem. 5.2.1). This h-principle opens the possibility to extend Theorem 1.1.3 to more general interfaces as was done in [22, 70, 113, 27] for the case of constant viscosity. We plan to explore it in future works.

For the rest of this section we consider the case of constant viscosity $\mu_h = \mu_l$.

The Muskat problem

The investigations on the Muskat problem ([110]) which deals with the interface evolution under the assumption of immiscibility, have been very intense both in the applied community due to the many applications (see e.g. [134, 131, 80, 97]) and in the theoretical side as this constitutes a challenging free boundary problem.

Mathematically, the theory has bifurcated into two regimes, the so-called stable regime and unstable regime. This division arises from the linear stability analysis of the equation for the interface evolution. It is classical (see e.g. [37]) that such linear stability is characterized by the sign of the Rayleigh-Taylor function. This simply classifies whether the heavier fluid remains (locally) below the lighter one or not.

The instability in the linearization is called **Rayleigh-Taylor** (or Saffman-Taylor [119]) for the Muskat problem. In the graph case, it corresponds to the heavier fluid above the lighter one, what is called the fully unstable regime. In this case, the problem is ill-posed unless the initial data is real-analytic (see e.g. [123]). However, practical and numerical experiments show the existence of the so-called mixing solutions, solutions in which there exists a mixing zone where the two fluids mix stochastically (see e.g. [134, 80]). Numerically, it can be seen that small disturbances of an analytic initial interface increases rapidly creating finger patterns at different scales in the unstable region (see e.g. [131, 97]).

In spite of the fact that the interface evolution is ill-posed and in accordance with what is observed in the experiments, weak solutions to IPM, in the fully unstable case, have been constructed in the last years by replacing the continuum free boundary assumption with the opening of a mixing zone Ω_{mix} where the fluids begin to mix wildly. These mixing solutions are recovered by the convex integration method applied in Ω_{mix} to a subsolution, which is intended to be a kind of coarse-grained solution to IPM. As we mentioned, the subsolutions are very related to the Lagrangian relaxed solutions of Otto [116, 115] (see also [83]).

A striking result from [24, 23] shows that there exist analytic initial interfaces in the fully stable regime (i.e. a graph) such that part of the curve turns to the unstable regime (i.e. no longer a graph) and later, at some $T_* > 0$, the interface $z(T_*)$ is analytic but at a point in the unstable region where it is not C^4. The argument in [23] could be adapted to prove weaker singularities in C^k where $k \geq 5$ (i.e. the interface leaves to be C^k but is still C^{k-1}). Thus, the Rayleigh-Taylor instability can arise spontaneously and the regularity might break down. After the blow-up time T_* it is to be expected that the Muskat problem is ill-posed.

In this dissertation we give a method to construct mixing solutions to IPM in the Muskat partially unstable case. The original motivation was to continue the solutions after the breakdown described in the previous paragraph. However, there are numerous scenarios which are partially unstable. In this dissertation we will concentrate on two of them: The so-called **bubble interfaces** where the two fluids are separated by a closed chord-arc curve (see [73] for the case with surface tension) and the **turned interfaces** where the interface is an open chord-arc curve which cannot be parametrized as a graph. We describe both scenarios readily, prior to the statement of the theorems. In both cases the initial density will be written as (recall $\rho_h > \rho_l$)

$$\rho^\circ(x) := \begin{cases} \rho_l, & x \in \Omega_l^\circ, \\ \rho_h, & x \in \Omega_h^\circ. \end{cases} \tag{1.18}$$

The bubble type initial interfaces are described by

$$\begin{aligned} \Omega_l^\circ &\equiv \text{ exterior domain of } z^\circ, \\ \Omega_h^\circ &\equiv \text{ interior domain of } z^\circ, \end{aligned} \tag{1.19}$$

for some closed chord-arc curve $z^\circ \in H^k(\mathbb{T};\mathbb{R}^2)$ with k big enough.

The turned type initial interfaces are described by

$$\begin{aligned} \Omega_l^\circ &\equiv \text{ upper domain of } z^\circ, \\ \Omega_h^\circ &\equiv \text{ lower domain of } z^\circ, \end{aligned} \tag{1.20}$$

for some open chord-arc curve z° whose turned region $\{\partial_\alpha z_1^\circ(\alpha) \leq 0\}$ has positive measure. Here we consider both the x_1-periodic case $z^\circ - (\alpha, 0) \in H^k(\mathbb{T};\mathbb{R}^2)$ and the asymptotically flat case $z^\circ - (\alpha, 0) \in H^k(\mathbb{R};\mathbb{R}^2)$ with k big enough.

Now we are ready to state our two main theorems, which appear in [27], joint work with Ángel Castro and Daniel Faraco.

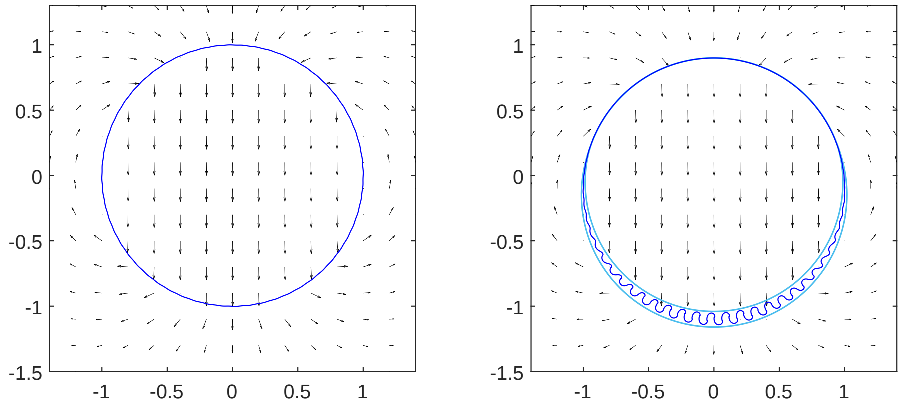
Theorem 1.1.4. *For every closed chord-arc curve $z^\circ \in H^6(\mathbb{T};\mathbb{R}^2)$ there exist infinitely many mixing solutions to IPM starting from* (1.18) *and* (1.19)*.*

Theorem 1.1.5. *For every open chord-arc curve z°, either x_1-periodic $z^\circ - (\alpha, 0) \in H^6(\mathbb{T};\mathbb{R}^2)$ or asymptotically flat $z^\circ - (\alpha, 0) \in H^6(\mathbb{R};\mathbb{R}^2)$, whose turned region $\{\partial_\alpha z_1^\circ(\alpha) \leq 0\}$ has positive measure there exist infinitely many mixing solutions to IPM starting from* (1.18) *and* (1.20)*.*

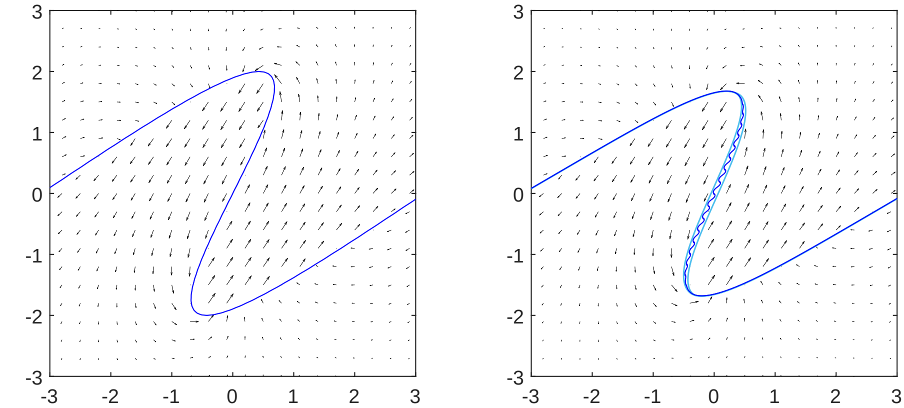
As in [126, 22, 70, 113], our mixing zone grows linearly in time around an evolving pseudo-interface. However, in Theorems 1.1.4 and 1.1.5 the mixing region must be localized in a neighborhood of the unstable region. Furthermore, this approach reveals the admissible regime for the growth-rate $c(\alpha)$ of the mixing zone compatible with the relaxation of IPM. This is

$$\left| c(\alpha) + \frac{\sigma(\alpha)}{\sqrt{\sigma(\alpha)^2 + \varpi(\alpha)^2}} \right| < 1 \quad \text{on} \quad c(\alpha) > 0, \tag{1.21}$$

which is characterized by the Rayleigh-Taylor function σ and the vorticity strength ϖ. Observe that (1.21) prevents the two fluids from mixing wherever the initial interface is stable ($\sigma(\alpha) > 0$) and there is no vorticity ($\varpi(\alpha) = 0$).



(a) A bubble type initial interface. (b) The localized mixing zone.



(c) A turned type initial interface. (d) The localized mixing zone.

Figure 1.2: (a)(c) The initial interface $z^{\circ}(\alpha)$ separating two fluids with different constant densities $\rho_{\pm} = \pm 1$ as in (1.19)(1.20) respectively. (b)(d) At some $t > 0$, the two boundaries of the non-mixing zones $z_{\pm}(t) = z(t) \mp tc\tau^{\perp}$ (light blue) for some pseudo-interface $z(t, \alpha)$ and growth-rate $c(\alpha)$, with $\tau(\alpha) = \frac{\partial_\alpha z^{\circ}(\alpha)}{|\partial_\alpha z^{\circ}(\alpha)|}$. Inside $\Omega_{\text{mix}}(t)$ we plot the Rayleigh-Taylor instability (dark blue). In all the figures we have added the velocity field $v(t)$ outside Ω_{mix}.

1.2 Convex integration method

The concept of h-principle (homotopy principle) and the convex integration method was developed in Differential Geometry by Gromov ([78]) as a far-reaching generalization of the groundbreaking work of Nash ([111]) and Kuiper ([88]) for isometric embeddings. Remarkably, Müller and Šverak ([108]) combined this method with Tartar compensated compactness ([129]) to apply it to PDEs and Calculus of Variations (see also [45, 84]).

The proofs of the theorems presented in this dissertation rely on the pioneering adaptation of the convex integration method to Hydrodynamics by De Lellis and Székelyhidi ([54, 53]). The method has turned out to be very robust and flexible and the research on it has been extremely intense in the last decade. We contempt ourselves with describing a few landmarks: It has successfully described several problems related to turbulence as the Onsager's conjecture (see e.g. [81, 13, 47]), the evolution of active scalars ([39, 122, 85, 82, 14, 79]) and transport equations ([43, 106, 105, 104]), the compressible Euler equations (see e.g. [30, 29, 99, 68, 1, 98]), the Navier-Stokes equations (see e.g. [16, 12, 31, 17]) and Magnetohydrodynamics ([65, 9, 66]) (see also the surveys [55, 52, 15] and the references therein).

The philosophy of the convex integration method consists of adding suitable corrections to switch from a subsolution $\bar{u}$, which is intended to be a kind of macroscopic solution, to exact solutions u. In the problems we consider the system of PDEs can be rewritten as a linear partial differential system coupled with a non-linear pointwise constraint ([54, 39]). Thus, $\bar{u}$ solves the same linear system of PDEs while satisfying a relaxation of the pointwise constraint for u. In the Tartar framework, the relaxation is determined by the compatibility of the constraint K with the wave cone Λ, which is expressed in terms of $K^\Lambda \equiv \Lambda$-convex hull of K. However, when the explicit computation of K^Λ is unattainable due to the high complexity and dimensionality, it is more practical to consider a simpler but still large enough relaxation. When these correcting terms can be constructed and the relaxation satisfies some geometric and functional properties, the convex integration method yields an h-principle whereby the problem of finding exact solutions is reduced to find a subsolution. Schematically,

$$\begin{array}{ccccc} & \substack{\text{microscopic}\\ \text{problem}} & \xrightarrow{\text{relaxation}} & \substack{\text{macroscopic}\\ \text{problem}} & \\ & \downarrow & \text{h-principle} & \downarrow & \\ \text{solution} & u & \xleftarrow[\substack{\text{convex}\\ \text{integration}}]{} & \bar{u} & \text{subsolution} \end{array} \tag{1.22}$$

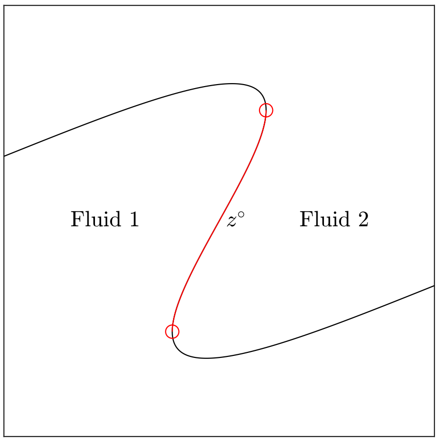
Turbulence zone

Although unstable configurations in Hydrodynamics are very difficult to model, De Lellis-Székelyhidi's version of convex integration have successfully describe several examples as the Rayleigh-Taylor instability for IPM ([126, 22, 70, 113]), and the Kelvin-Helmholtz ([127]) and Rayleigh-Taylor ([76, 75]) instabilities for the incompressible Euler equations.

As we mentioned in the previous section, in this dissertation we investigate the scope of this viewpoint to the Kelvin-Helmholtz instability for the incompressible Euler equation and the Rayleigh-Taylor instability for IPM in several scenarios which were not achieved before.

One of them is the case of IPM with viscosity jump, for which we compute explicitly the Λ-lamination hull of the corresponding constraint (see [126] for the case of constant viscosity). The switch from $\mu_h = \mu_l$ to $\mu_h \neq \mu_l$, which originally looks innocent, turns the relation between the components of the subsolution less explicit, which ends up hampering considerably the proof of the hypothesis required for the h-principle (see chapter 5 for further details). As we mentioned in section 1.1.2, this h-principle allows to construct weak solutions to IPM starting from the unstable planar phase (1.17) and displaying a turbulent mixing behavior inside a strip, the mixing zone, which grows linearly in time around the interface $z^\circ(\alpha) = (\alpha, 0)$.

More generally, we would like to consider two fluids which are smooth outside an interface z° separating them at $t = 0$. Let us assume that there is an unstable region where the interface evolution is expected to break. Then, for later times it seems natural that this unstable region will be surrounded by a turbulence zone $\mathscr{U}(t)$ whose boundary curves $z_+(t)$ and $z_-(t)$ collapse in a single interface $z(t)$ inside the stable region (see Figure 1.3). More precisely, $\mathscr{U}(t)$ surrounds a pseudo-interface $z(t)$ which evolves from z°. Outside the turbulence zone and the interface the fluids would remain smooth, but inside $\mathscr{U}$ it would be observed a turbulent behavior. Although the mathematical description of these turbulent flows seems out of reach because of its unpredictable nature, the convex integration method allows to recover them by adding highly oscillatory terms to a subsolution. This subsolution is indeed an exact solution outside the turbulence zone (and smooth outside the interface), but inside $\mathscr{U}$ it is only required that it solves the relaxed problem with certain regularity.



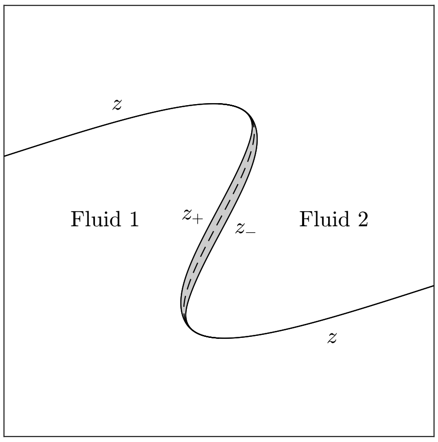


Figure 1.3: Left: The initial interface z° and its unstable region (red). Right: The turbulence zone $\mathscr{U}(t)$ (gray) surrounding the pseudo-interface $z(t)$.

In the case of vortex sheet data for the incompressible Euler equation, in general terms, the full interface is unstable due to the vorticity strength. As we mentioned in section 1.1.1, following this approach we construct, jointly with László Székelyhidi, weak solutions dissipating the kinetic energy inside the turbulence zone. In this case, $\mathscr{U}$ is denoted by Ω_{tur}.

In the case of Muskat data for IPM in the partially unstable regime, the region where the heavier fluid is above the lighter one is unstable (this is precisely Figure 1.3 if Fluid 1 is the heavier). As we mentioned in section 1.1.2, following this approach we construct, jointly with Ángel Castro and Daniel Faraco, weak solutions whose mixing zone is localized in a neighborhood of the unstable region. In this case, $\mathscr{U}$ is denoted by Ω_{mix}. This is the first result that shows the compatibility between the theory of interface evolution and the convex integration method.

Quantitative h-principle

The h-principle we follow is based on the version presented in [53] for the incompressible Euler equation, which yields weak solutions in the class $C_tL^\infty_{w^*}$. In chapter 3 we present a quantitative version for a class of evolution equations with the aim of recovering more information from the subsolution. Although this result is technical and will be presented rigorously in Theorems 3.3.1-3.3.3, let us write here an informal version.

Theorem 1.2.1. *Assuming that the "macroscopic problem" satisfies certain hypothesis, there exist infinitely many weak solutions to the "microscopic problem". Moreover, the solutions equal the subsolution outside $\mathscr{U}$ but can behave wildly inside $\mathscr{U}$. However, there exist infinitely many weak solutions whose macroscopic behavior inside $\mathscr{U}$ essentially agrees with the subsolution.*

In the case of IPM, this quantitative h-principle allows to construct weak solutions $(\theta, v) \in C_tL^\infty_{w^*}$ such that, at each $0 < t \leq T$, the space is split into three complementary domains, $\Omega_h(t)$, $\Omega_l(t)$ and $\Omega_{\text{mix}}(t)$, the two first are the non-mixing zones

$$\theta(t,x) = \begin{cases} +1, & x \in \Omega_h(t), \\ -1, & x \in \Omega_l(t), \end{cases} \tag{1.23}$$

while it behaves wildly inside Ω_{mix}

$$\int_\Omega (1-\theta(t,x)^2)\,\mathrm{d}x = 0 < \int_\Omega (1-\theta(t,x))\,\mathrm{d}x \int_\Omega (1+\theta(t,x))\,\mathrm{d}x, \tag{1.24}$$

for every open $\emptyset \neq \Omega \subset \Omega_{\text{mix}}(t)$.

The property (1.24) was first introduced in [22] on space-time balls. In [26] we took care of replacing "space-time" by the stronger and more suitable version "space at each time slice". This property can be read as the phase (and so the density and the viscosity) takes values wildly between $+1$ and -1 (resp. a_h and a_l for $a = \rho, \mu$) inside the mixing zone, thus justifying its name. Both (1.23) and (1.24) correspond to the microscopic part of the above h-principle.

The property (1.24) predicts mixing in every open domain inside Ω_{mix}, but it does not give information about the volume proportion of each fluid. As it stands it does not exclude that arbitrarily close to Ω_h could be a sufficiently big ball with 99% of ρ_l. In spite of the stochastic nature of the mixing phenomenon, this is obviously unrealistic from the experiments.

As we will explain in more detail in chapter 5 (see also [26]), we find natural to obtain mixing solutions displaying a degraded macroscopic behavior inside Ω_{mix}. For the case of constant viscosity, the subsolution that naturally appears from the unstable planar phase was found in [126] by Székelyhidi. Remarkably, Székelyhidi's subsolution $\bar\theta$ agrees (up to rescaling in time: $\bar\theta(t) = \Theta(\alpha t)$ for some $0 < \alpha < 1$) with the Lagrangian relaxed solution of Otto

$$\Theta(t,x) := \begin{cases} +1, & x_2 > +2t, \\ \dfrac{x_2}{2t}, & |x_2| \leq 2t, \\ -1, & x_2 < -2t, \end{cases} \tag{1.25}$$

which aims to capture the macroscopic properties of exact solutions to IPM, thus giving a prediction of the actual shape and evolution of the mixing profile.

This motivates to look for solutions displaying a linearly degraded macroscopic behavior. However, an error in the average between the solutions θ and the subsolution $\bar{\theta}$ is unavoidable on sufficiently small regular domains due to the Lebesgue differentiation theorem. Since this error spreads as the fluids advance into the mixing zone, it must depend also on the distance to where the fluids begin to mix. In [26] we construct such linearly degraded mixing solutions. For some fixed (distance) function $D \in C(\Omega_{\text{mix}};(0,1])$ and $\gamma \in [0,1)$, they satisfy

$$\left|\fint_R (\theta-\bar{\theta})(t,x)\,\mathrm{d}x\right| \leq \frac{1\wedge|R|^\gamma}{|R|}D(t,\langle R\rangle), \tag{1.26}$$

for every rectangle $R \subset \Omega_{\text{mix}}(t)$ and $0 < t \leq T$, where $a \wedge b \equiv \min\{a,b\}$ and $\langle R\rangle \equiv \fint_R x\,\mathrm{d}x = \frac{1}{|R|}\int_R x\,\mathrm{d}x$ is the center of mass of R.

The property (1.26) corresponds to the macroscopic part of the above h-principle. Observe that the volume proportion of fluid with phase $\pm$ in R is

$$\frac{|\{x \in R\,:\,\theta(t,x)=\pm 1\}|}{|R|} = \frac{1}{2}\left(1 \pm \fint_R \theta(t,x)\,\mathrm{d}x\right), \tag{1.27}$$

that is, the average of θ quantifies the amount of each fluid. From [126] we know the existence of a sequence of solutions θ_k satisfying $\theta_k \overset{*}{\rightharpoonup} \bar{\theta}$. Thus, we would like to obtain solutions which are as close as possible to satisfy

$$\fint_R \theta(t,x)\,\mathrm{d}x \approx \fint_R \bar{\theta}(t,x)\,\mathrm{d}x = \fint_L x_2\,\mathrm{d}x_2 = \langle L\rangle \in (-1,1), \tag{1.28}$$

for every rectangle $R = S \times 2\alpha t L \subset \Omega_{\text{mix}}(t)$ and $0 < t \leq T$. However, Lebesgue differentiation theorem tells us that

$$\lim_{\substack{R\to\{x_0\}\\ \text{regular}}} \fint_R \theta(t,x)\,\mathrm{d}x = \theta(t,x_0), \tag{1.29}$$

for almost every x_0 and all t, where recall θ jumps unpredictably between $+1$ and -1 because of (1.24). In other words, if the position is localized, $R \to \{x_0\}$, then the average of θ is undetermined from the subsolution. The other side of the coin is given by (1.26) because it states that we can know exactly the average of θ on unbounded domains. Schematically, this phenomenon can be interpreted as an "uncertainty principle" as follows

		R	$\fint_R \theta(t,x)\,\mathrm{d}x$
Certainty	Position	$\{x_0\}$	unpredictably
	Average	unbounded	$\langle L\rangle$

As we have already commented, the first row is nothing but (1.24) in combination with (1.29). The second row is due to (1.26) because these degraded mixing solutions satisfy

$$\lim_{\substack{L\to L_0\\ |S|\cdot|L|\to\infty}} \fint_R \theta(t,x)\,\mathrm{d}x = \langle L_0\rangle,$$

for every interval $L_0 \subset (-1,1)$ at each $t \in (0,T]$. Therefore, the volume proportion of fluid with phase $\pm$ in the strip $\mathbb{R} \times 2\alpha t L_0$ is exactly $\frac{1}{2}(1 \pm \langle L_0\rangle)$. For $L_0 = \{\frac{x_2}{2\alpha t}\}$, this exhibits a perfect linearly degraded macroscopic behavior on contour lines (cf. Remark 5.2.2).

Furthermore, (1.26) does not only quantify these extremal situations, $R \to \{x_0\}$ and $R \to$ unbounded, but also the intermediate cases. More precisely, for every small $\varepsilon > 0$, these degraded mixing solutions improve the knowledge of (1.27) around each point $x_0 \in \Omega_{\text{mix}}(t)$ and $0 < t \leq T$

$$\left| \fint_R \theta(t,x)\,\mathrm{d}x - \langle L \rangle \right| \leq \varepsilon, \tag{1.30}$$

for every rectangle $R \subset \Omega_{\text{mix}}(t)$ containing x_0 in the regime

$$\frac{1 \wedge |R|^\gamma}{|R|} D(t, \langle R \rangle) \leq \varepsilon.$$

Observe that $\gamma = 1$ is excluded (otherwise we may consider $|R| \to 0$). Thus, the uncertainty depends on the size of the rectangles and the distance to the (space-time) boundary of the mixing zone. In particular, the linearly degraded macroscopic behavior is almost perfect close to where the fluids begin to mix.

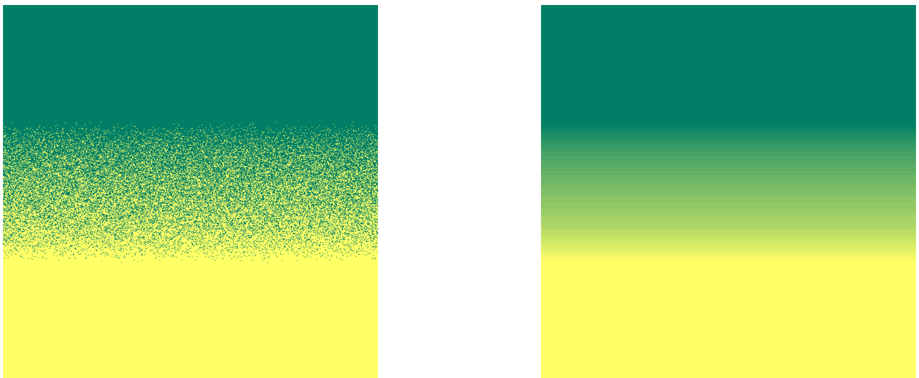


Figure 1.4: Left: The solution. Right: The subsolution.

In chapter 5 we extend these results to the case of viscosity jump ([102]). For the case of constant viscosity, Castro, Córdoba and Faraco generalized Székelyhidi's result for non-flat interfaces in [22]. In this work the subsolution is a linear interpolation between $+1$ and -1 as in (1.25) inside the mixing zone. After that, we proved in [27] that the property (1.26) and its consequences hold by averaging on rectangles adapted to the evolution of the mixing zone.

In the case of the incompressible Euler equation, the microscopic part of Theorem 1.2.1 shows that, among the solutions obtained through convex integration, there are Euler velocities that are almost nowhere Hölder continuous inside Ω_{tur}. In spite of this, the macroscopic part of the theorem allows to select those which essentially behave like the macroscopic velocity.

Remark 1.2.1. We conclude mentioning that the microscopic part of Theorem 1.2.1 only shows that there exist solutions that behave really wild inside $\mathscr{U}$, but does not exclude the existence of other solutions with better regularity. In fact, it would be very interesting to explore the construction of more regular solutions inside the turbulence zone. Inspired by the Onsager's conjecture and the recent results obtained in this context, we would like to understand and achieve the regularity threshold inside the turbulence zone. We plan to explore this question in future works.

1.3 Notation

Complex coordinates for 2D flows

It will be very useful to identify the euclidean space $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ as usual

$$x = (x_1, x_2) \equiv x_1 + ix_2,$$

where $i \equiv (0,1)$ will play the role both of the imaginary unit and the standard vertical vector. Therefore, along the whole dissertation we will use complex coordinates and subindexes $1,2$ indicate real and imaginary parts for a complex number. We will denote

- $x^* := (x_1, -x_2) = x_1 - ix_2$.
- $x^\perp := (-x_2, x_1) = ix$.
- $x \cdot y := x_1y_1 + x_2y_2 = (xy^*)_1$.
- $x \cdot y^\perp := x_2y_1 - x_1y_2 = (xy^*)_2$.

In this setting, we have $\nabla = (\partial_1, \partial_2) = \partial_1 + i\partial_2$, and also $\nabla^* = \partial_1 - i\partial_2$ and $\nabla^\perp = i\nabla$.

Function spaces

- For $1 \leq p \leq \infty$, L^p denotes the Lebesgue space of p-integrable (or essentially bounded for $p = \infty$) functions with norm $\|\cdot\|_{L^p}$. For $1 < p \leq \infty$, we will denote $L^p_{w^*}$ to indicate that L^p is endowed with the weak*-topology.
- For $k \in \mathbb{N}_0$ and $0 < \delta \leq 1$, $C^{k,\delta}$ denotes the Hölder space with norm

$$\|f\|_{C^{k,\delta}} := \sup_{j\leq k} \|\partial^j f\|_{L^\infty} + |\partial^k f|_{C^\delta} \quad \text{with} \quad |g|_{C^\delta} := \sup_{\alpha,\beta} \frac{|g(\alpha) - g(\alpha-\beta)|}{|\beta|^\delta},$$

  and H^k the Hilbert space with norm

$$\|f\|_{H^k} := \left(\sum_{j=0}^{k} \|\partial^j f\|_{L^2}^2\right)^{\frac{1}{2}}.$$

Increments, quotients and Taylor series

Given $f : \mathbb{T} \to \mathbb{C}$ we denote

$$\delta_\beta f(\alpha) := f(\alpha) - f(\alpha - \beta),$$

and also

$$\Delta_\beta f(\alpha) := \frac{\delta_\beta f(\alpha)}{\beta}.$$

More generally, if f is k-times differentiable at α we denote its Taylor polynomial of order k and the corresponding reminder as

$$T^k_\beta f(\alpha) := \sum_{j=0}^{k} \frac{\partial^j_\alpha f(\alpha)}{j!}(-\beta)^j,$$

$$\Delta^{k+1}_\beta f(\alpha) := \frac{T^k_\beta f(\alpha) - f(\alpha-\beta)}{\beta^{k+1}} = \frac{\Delta^k_\beta f(\alpha) + \frac{(-1)^k}{k!}\partial^k_\alpha f(\alpha)}{\beta}.$$

In particular, if $f \in C^{k,\delta}(\mathbb{T};\mathbb{C})$ we have

$$\|\Delta^{k+1}_\beta f\|_{L^\infty} \leq \frac{|\partial^k_\alpha f|_{C^\delta}}{\binom{k+\delta}{k}k!}|\beta|^{\delta-1},$$

and if $f \in C^{k+1}(\mathbb{T};\mathbb{C})$ also

$$\Delta^{k+1}_0 f(\alpha) := \lim_{\beta\to 0} \Delta^{k+1}_\beta f(\alpha) = \tfrac{(-1)^k}{(k+1)!}\partial^{k+1}_\alpha f(\alpha).$$

Moreover, for $f \in C^k(\mathbb{T};\mathbb{C})$, the following decomposition will be useful

$$\Delta_\beta f(\alpha) = \sum_{j=1}^{k} \frac{\partial^j_\alpha f(\alpha)}{j!}(-\beta)^{j-1} + \beta^k \Delta^{k+1}_\beta f(\alpha).$$

Furthermore, given a function $f = f(\alpha)$ and another parameter β, it will be handy to use the expressions $f' = f(\alpha - \beta)$ as in [38], and thus $\delta_\beta f = f - f'$.

Similarly, given a time-dependent function f, we will denote

$$f^{(n)} := \partial^n_t f(0),$$

and

$$f^{n)} := \sum_{k=0}^{n} \frac{f^{(k)}}{k!}t^k, \qquad f^{(n+1} := \frac{f - f^{n)}}{t^{n+1}} = \frac{f^{(n} - \frac{1}{n!}f^{(n)}}{t}.$$

Chapter 2

Unstable interfaces and turbulence zone

The two problems we consider, the vortex sheet and the Muskat problems, have some common aspects that we have preferred to gather in this section. In both cases the fluid is incompressible ($\mathrm{div} v = 0$) and the vorticity ($\mathrm{rot} v = \omega$) will be a Radon measure ($\omega \in \mathcal{M}(\mathbb{R}^2)$) concentrated on the turbulence zone. Thus, the velocity v is recovered from the vorticity ω through the Biot-Savart law. For simplicity we consider in this section that ω is compactly supported ($\omega \in \mathcal{M}_c(\mathbb{R}^2)$) and that $\mathscr{D} = \mathbb{R}^2$.

Proposition 2.0.1 (Biot-Savart law)**.** *Let $\omega \in \mathcal{M}_c(\mathbb{R}^2)$. Any distributional solution v to*

$$\mathrm{div} v = 0, \qquad \mathrm{rot} v = \omega, \tag{2.1}$$

has the form $v = (K * \omega + f)^*$ for some entire function f, where K is the Cauchy kernel

$$K(x) := \frac{1}{2\pi i x},$$

and $K * \omega$ is defined in $L^1_{\mathrm{loc}}(\mathbb{R}^2)$ as

$$(K * \omega)(x) = \frac{1}{2\pi i} \int_{\mathbb{R}^2} \frac{1}{x - x'} \, \mathrm{d}\omega(x').$$

Furthermore, $K * \omega$ is holomorphic outside $\mathrm{supp}\, \omega$ *with decay*

$$(K * \omega)(x) = \frac{1}{2\pi i x}(\omega(\mathbb{R}^2) + O(|x|^{-1})), \qquad |x| \gg 1. \tag{2.2}$$

Proof. The proof is classical but we sketch it here for completeness. First of all, it is well known that $\bar\nabla K = -\delta_0$, that is,

$$\mathrm{div}(K^*) = 0, \qquad \mathrm{rot}(K^*) = \delta_0,$$

in the sense of distributions. Secondly, it is easy to check that $K * \omega$ is well defined in L^1_{loc}. Hence, it follows that $(K * \omega)^* = K^* * \omega$ satisfies (2.1). The decay property (2.2) follows from

$$(K * \omega)(x) - \frac{1}{2\pi i x}\omega(\mathbb{R}^2) = \frac{1}{2\pi i x} \int_{\mathbb{R}^2} \frac{x'}{x - x'} \, \mathrm{d}\omega(x').$$

Finally, observe that any $f = v^* - K * \omega$ must satisfy the Cauchy-Riemann equations. $\square$

For the initial value problems we consider the vorticity is initially concentrated on the interface $z^\circ(\alpha)$ with some vorticity strength $\varpi^\circ(\alpha)$, that is $\omega^\circ = \varpi^\circ\delta_{z^\circ}$ (the pushforward measure $z^{\circ\sharp}(\varpi^\circ\,\mathrm{d}\beta)$) which is expressed in the sense of distributions as

$$\langle\omega^\circ,\psi\rangle = \int \psi^\circ(\beta)\varphi(z^\circ(\beta))\,\mathrm{d}\beta, \qquad \psi\in C_c^\infty(\mathbb{R}^2).$$

By the Biot-Savart law, the initial velocity is given by

$$v^\circ(x)^* = \frac{1}{2\pi i}\int \frac{\varpi^\circ(\beta)}{x-z^\circ(\beta)}\,\mathrm{d}\beta, \qquad x\neq z^\circ(\beta).$$

Both the classical vortex sheet and Muskat problems assume that the fluids remain irrotational outside some evolving interface $z(t,\alpha)$ with some vorticity strength $\varpi(t,\alpha)$, that is,

$$\omega(t) = \varpi(t)\delta_{z(t)}.$$

Hence, for later times

$$v(t,x)^* = \frac{1}{2\pi i}\int \frac{\varpi(t,\beta)}{x-z(t,\beta)}\,\mathrm{d}\beta, \qquad x\neq z(t,\beta).$$

We recall the classical Sokhotski–Plemelj theorem (the definition of chord-arc curves can be found in Def. 2.1.1). The proof follows similarly to section 2.2 (see also [69, chapter 3]).

Proposition 2.0.2 (Sokhotski–Plemelj theorem)**.** *Let $\omega = \varpi\delta_z$ with $\varpi\in C^{0,\delta}$ and $z\in C^{1,\delta}$ be oriented, closed and chord-arc. Let*

$$\Omega_+ \equiv \textit{domain to the left side of } z$$
$$\Omega_- \equiv \textit{domain to the right side of } z.$$

Then,

$$\lim_{\Omega_\pm\ni x\to z(\alpha)}(K*\omega)(x) = \mathscr{B}(\omega)(\alpha)^* \mp \frac{1}{2}\frac{\varpi(\alpha)}{\partial_\alpha z(\alpha)},$$

*where $\mathscr{B}(\omega)$ is the **Birkhoff-Rott** operator (*pv $\equiv$ *Cauchy principal value)*

$$\mathscr{B}(\omega)(\alpha)^* := \frac{1}{2\pi i}\mathrm{pv}\int \frac{\varpi(\beta)}{z(\alpha)-z(\beta)}\,\mathrm{d}\beta.$$

As a result, $v^* = K*\omega$ is uniformly bounded in $\mathbb{R}^2$, holomorphic outside z but discontinuous along it. However, the normal component of v along z is well defined, namely

$$\lim_{\Omega_\pm\ni x\to z(\alpha)}(v(x)-\mathscr{B}(\omega)(\alpha))\cdot\partial_\alpha z(\alpha)^\perp = 0.$$

The evolution of z is driven by $\mathscr{B}$ according to

$$(\partial_t z - \mathscr{B}(\omega))\cdot\partial_\alpha z^\perp = 0,$$

with $\omega(z) = \varpi(z)\delta_z$, namely $\varpi = \varpi^\circ$ for the vortex sheet problem and $\varpi = (\rho_+ - \rho_-)\partial_\alpha z_2$ for the Muskat problem. This Cauchy problem for the interface z has been widely studied in the literature.

However, as we mentioned in chapter 1, in this dissertation we consider several scenarios for which this Cauchy problem for z is ill-posed. In such case, we will replace the continuum free boundary assumption with the opening of a turbulence zone $\mathscr{U}$ where the vorticity will be concentrated. In the case of the vortex sheet problem this is denoted by Ω_{tur} and in the case of the Muskat problem this is called the mixing zone Ω_{mix}.

2.1 Turbulence zone

At each time slice $0 < t \leq T \ll 1$, the turbulence zone is the open set in $\mathbb{R}^2$ given by

$$\mathscr{U}(t) := \{Z(t,\alpha,\lambda) \,:\, c(\alpha) > 0,\ \lambda \in (-1,1)\},$$

parametrized by the map

$$Z(t,\alpha,\lambda) := z(t,\alpha) + \lambda t c(\alpha)\tau(\alpha)^{\perp}, \tag{2.3}$$

where $\tau(\alpha)$ is an unitary vector field (tangential to $z^\circ(\alpha)$), $c(\alpha) \geq 0$ is the growth-rate of the turbulence zone and $z(t,\alpha)$ is the (oriented) pseudo-interface evolving from $z^\circ(\alpha)$, that we have to determine.

For any fixed $\lambda \in [-1,1]$, we will denote

$$z_\lambda := Z(\cdot,\lambda).$$

Assuming that $\partial_\alpha z \cdot \tau > 0$, we define

$$\begin{aligned} \Omega_+(t) &\equiv \text{ domain to the left side of } z_+(t), \\ \Omega_-(t) &\equiv \text{ domain to the right side of } z_-(t). \end{aligned} \tag{2.4}$$

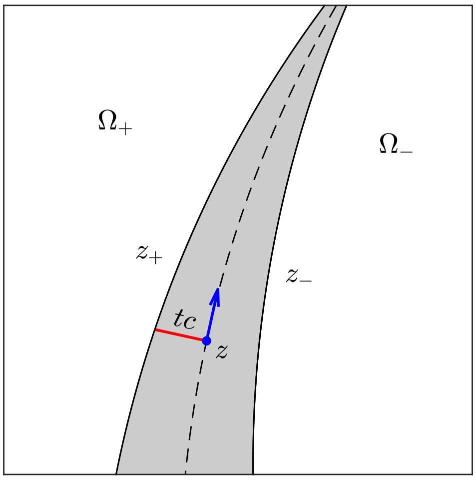


Figure 2.1: The turbulence zone $\mathscr{U}(t)$ grows linearly in time with growth rate $c(\alpha)$ around the pseudo-interface $z(t,\alpha)$.

Remark 2.1.1. Although we may consider more general growths for the turbulence zone (see [76, 75] where the growth is of order t^2) for the problems we consider in this dissertation $tc(\alpha)$ is adequate for short times. We remark that the mixing zone in chapter 5 surrounds $z^\circ(\alpha) = (\alpha, 0)$ in the x_2-interval $(-(1+\frac{\mu_h}{\mu_l})t, (1+\frac{\mu_l}{\mu_h})t)$, and thus it does not grow in a symmetrical fashion w.r.t. z° for the viscosity jump case $\mu_h \neq \mu_l$.

Although we will take τ as the tangent field to z° in chapter 4, we will keep it general here. In chapter 6 it will be convenient to replace τ by $-\tau$, but this requires minor modifications.

2.1.1 Geometric setup

For simplicity we will focus on closed curves in this chapter. Let us assume w.l.o.g. that z° is positively oriented ($\circlearrowleft$), and also that z° is the arc-length ($|\partial_\alpha z^\circ| = 1$) parametrization. Thus, $\mathbb{T} = [-\ell_\circ/2, \ell_\circ/2]$ where $\ell_\circ = \text{length}(z^\circ)$.

Apart from regularity assumptions, the curve z and the unitary vector field τ needs to satisfy an angle and a chord-arc condition in a uniform manner. Prior to state the assumptions, let us introduce the angle constant of z w.r.t. τ

$$\mathcal{A}(z) := \inf_\alpha \frac{\partial_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|} \cdot \tau(\alpha), \tag{2.5}$$

and the chord-arc constant of z

$$\mathcal{C}(z) := \sup_{\alpha,\beta} \left| \frac{\beta}{z(\alpha) - z(\alpha - \beta)} \right|. \tag{2.6}$$

Definition 2.1.1. A continuous curve z is called **chord-arc** if $\mathcal{C}(z) < \infty$.

The chord-arc condition is usually imposed when considering Birkhoff-Rott type operators (cf. [135, sec. 1.1]) because it avoids self-intersections and bad parametrizations. Moreover, it gives a lower bound of the proximity of different points at z°: $|z^\circ(\alpha) - z^\circ(\alpha - \beta)| \geq |\beta|/\mathcal{C}(z^\circ)$, thus measuring the singularity in $\mathscr{B}$ at time $t = 0$. However, for $t > 0$ the corresponding Birkhoff-Rott operator requires to compare different points at the boundary of the turbulence zone.

Lemma 2.1.1. *Let $z \in C([0,T]; C^{1,\delta}(\mathbb{T};\mathbb{R}^2))$. Assume that, for some $0 < A, C, R < \infty$,*

$$\mathcal{A}(z(t)) > A, \qquad \mathcal{C}(z(t)) < C, \qquad |\partial_\alpha z(t)|_{C^\delta} < R,$$

for all $0 \leq t \leq T$. Then, there exists $0 < T'(A, C, R, \delta, \|c\tau\|_{C^{1,\delta}}) \leq T$ such that the following "equi-chord-arc condition" holds

$$|z_\lambda(t,\alpha) - z_\mu(t, \alpha - \beta)|^2 \geq D\left(\frac{\beta^2}{(2C)^2} + ((\lambda - \mu)tc(\alpha))^2\right), \tag{2.7}$$

for all $\alpha, \beta \in \mathbb{T}$, $\lambda, \mu \in [-1,1]$ and $0 \leq t \leq T'$, where $D \equiv 1 - \sqrt{1 - (A/2)^2}$. In addition,

$$\sup_{\lambda \in [-1,1]} \mathcal{A}(z_\lambda(t)) > A/2. \tag{2.8}$$

Proof. First of all notice that $A < 1$, and thus $D < 1$ as well. In particular, (2.7) holds for $t = 0$. Henceforth, let $0 < t \leq T'$ for some $0 < T' \leq T$ to be determined. Notice that we can take T' satisfying (recall sec. 1.3)

$$|\delta_\beta z_\mu| \geq |\delta_\beta z| - t|\delta_\beta(c\tau)| \geq \left(\frac{1}{C} - t|c\tau|_{C^1}\right)|\beta| \geq \frac{|\beta|}{2C}. \tag{2.9}$$

We split the proof into two cases, depending on the following parameter

$$r \equiv \left(\frac{A}{2^3 CR}\right)^{1/\delta}.$$

Case $|\beta|\leq r$. By writing,

$$z_\lambda - z'_\mu = \delta_\beta z_\mu + (\lambda-\mu)tc\tau^\perp, \tag{2.10}$$

we split the l.h.s. of (2.7) into

$$|z_\lambda - z'_\mu|^2 = |\delta_\beta z_\mu|^2 + ((\lambda-\mu)tc)^2 + 2(\lambda-\mu)tc\delta_\beta z_\mu\cdot\tau^\perp. \tag{2.11}$$

Let us analyze the third term. By our choice of r and using $|\partial_\alpha z|\geq 1/C$, we can take T' satisfying

$$\left|\frac{\delta_\beta z_\mu}{|\delta_\beta z_\mu|} - \frac{\partial_\alpha z}{|\partial_\alpha z|}\right| \leq 2\frac{|\Delta_\beta z_\mu - \partial_\alpha z|}{|\partial_\alpha z|} \leq 2C(|\partial_\alpha z|_{C^\delta}|\beta|^\delta + t|c\tau|_{C^1}) \leq A/2.$$

Then, by adding and subtracting $\partial_\alpha z/|\partial_\alpha z|$, we deduce that

$$\frac{|\delta_\beta z_\mu\cdot\tau|}{|\delta_\beta z_\mu|} \geq \frac{\partial_\alpha z}{|\partial_\alpha z|}\cdot\tau - \left|\frac{\delta_\beta z_\mu}{|\delta_\beta z_\mu|} - \frac{\partial_\alpha z}{|\partial_\alpha z|}\right| \geq A/2,$$

which implies that

$$(\delta_\beta z_\mu\cdot\tau^\perp)^2 = |\delta_\beta z_\mu|^2 - (\delta_\beta z_\mu\cdot\tau)^2 \leq (1-(A/2)^2)|\delta_\beta z_\mu|^2. \tag{2.12}$$

Finally, by applying (2.9) and (2.12) into (2.11), we deduce that

$$\begin{aligned}|z_\lambda - z'_\mu|^2 &\geq (1-\sqrt{1-(A/2)^2})(|\delta_\beta z_\mu|^2 + ((\lambda-\mu)tc)^2)\\ &\geq (1-\sqrt{1-(A/2)^2})\left(\frac{\beta^2}{(2C)^2} + ((\lambda-\mu)tc)^2\right).\end{aligned}$$

Case $|\beta|>r$. On the one hand, by applying (2.9)(2.10), the l.h.s. of (2.7) can be bounded from below as

$$|z_\lambda - z'_\mu| \geq |\delta_\beta z_\mu| - 2t\|c\|_{C^0} \geq \frac{|\beta|}{2C} - 2t\|c\|_{C^0}.$$

On the other hand, the r.h.s. of (2.7) can be bounded from above as

$$\frac{\beta^2}{(2C)^2} + ((\lambda-\mu)tc)^2 \leq \frac{\beta^2}{(2C)^2} + (2t\|c\|_{C^0})^2.$$

Thus, it is enough to guarantee that

$$\left(\frac{|\beta|}{2C} - 2t\|c\|_{C^0}\right)^2 \geq D\left(\frac{\beta^2}{(2C)^2} + (2t\|c\|_{C^0})^2\right),$$

or equivalently

$$\frac{(1-s)^2}{1+s^2} \geq D \quad \text{with} \quad s \equiv \frac{4C\|c\|_{C^0}}{|\beta|}t.$$

Since $D<1$, this holds for all $|\beta|>r$ by taking T' small enough.

Finally, it is clear that (2.8) holds for small times. □

Remark 2.1.2. The conditions (2.7)(2.8) imply that map $(\alpha,\lambda)\mapsto Z(t,\alpha,\lambda)$ is a diffeomorphism from $\{c(\alpha)>0\}\times(-1,1)$ to $\mathscr{U}(t)$ with Jacobian $tc(\partial_\alpha z_\lambda\cdot\tau)>0$.

2.2 The velocity

Both for the vortex sheet and the Muskat problems, we will consider that the vorticity of the subsolution will be concentrated on the boundary of the turbulence zone

$$\bar{\omega}(t) := \frac{1}{2}\sum_{b=\pm}\varpi_b(t)\delta_{z_b(t)}.$$

Thus, the macroscopic velocity $\bar{v}(t)$ is recovered from $\bar{\omega}(t)$ through the Biot-Savart law

$$\bar{v}(t,x)^* = \frac{1}{2}\sum_{b=\pm}\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{\varpi_b(t,\beta)}{x-z_b(t,\beta)}\,\mathrm{d}\beta, \qquad x\neq z_b(t,\beta).$$

Let us compute an alternative expression for $\bar{v}$ which will be helpful. Firstly, we define the **index** of $x_0\in\mathbb{R}^2\setminus z(\mathbb{T})$ w.r.t. z as

$$\mathrm{Ind}_z(x_0) := \frac{1}{2\pi i}\int_z\frac{\mathrm{d}x}{x-x_0} = \begin{cases}1, & x_0 \text{ inside } z,\\ 0, & x_0 \text{ outside } z.\end{cases}$$

Given $x\neq z_b(t,\beta)$, observe that we can write

$$\bar{v}(t,x)^* = \frac{1}{2}\sum_{b=\pm}\left(\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{\varpi_b(t,\beta)-w_b(t,x)\partial_\alpha z_b(t,\beta)}{x-z_b(t,\beta)}\,\mathrm{d}\beta - w_b(t,x)\mathrm{Ind}_{z_b(t)}(x)\right),$$

for any $w_b(t,x)\in\mathbb{R}^2$. Then, taking $w_b=\frac{\varpi_b(t,\alpha)}{\partial_\alpha z_b(t,\alpha)}$, we get

$$\begin{aligned}\bar{v}(t,x)^* = \frac{1}{2}\sum_{b=\pm}\Bigg(&\frac{1}{2\pi i}\frac{\varpi_b(t,\alpha)}{\partial_\alpha z_b(t,\alpha)}\int_{\mathbb{T}}\frac{\partial_\alpha z_b(t,\alpha)-\partial_\alpha z_b(t,\beta)}{x-z_b(t,\beta)}\,\mathrm{d}\beta\\ &-\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{\varpi_b(t,\alpha)-\varpi_b(t,\beta)}{x-z_b(t,\beta)}\,\mathrm{d}\beta - \frac{\varpi_b(t,\alpha)}{\partial_\alpha z_b(t,\alpha)}\mathrm{Ind}_{z_b(t)}(x)\Bigg),\end{aligned} \tag{2.13}$$

valid for any $0\leq t\leq T$, $x\neq z_b(t,\beta)$ and $\alpha\in\mathbb{T}$.

In particular, for $t=0$ we have

$$\begin{aligned}v^\circ(x)^* = &\frac{1}{2\pi i}\frac{\varpi^\circ(\alpha)}{\partial_\alpha z^\circ(\alpha)}\int_{\mathbb{T}}\frac{\partial_\alpha z^\circ(\alpha)-\partial_\alpha z^\circ(\beta)}{x-z^\circ(\beta)}\,\mathrm{d}\beta\\ &-\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{\varpi^\circ(\alpha)-\varpi^\circ(\beta)}{x-z^\circ(\beta)}\,\mathrm{d}\beta - \frac{\varpi^\circ(\alpha)}{\partial_\alpha z^\circ(\alpha)}(\alpha)\mathrm{Ind}_{z^\circ}(x).\end{aligned}$$

By considering nontangential limits ($x\to z^\circ(\alpha)$) we deduce that v° is bounded in a neighborhood of z°, and consequently $v^\circ = O((1+|x|)^{-1})$ in terms of $\|z^\circ\|_{C^{1,\delta}}$, $\|\varpi^\circ\|_{C^{0,\delta}}$ and $\mathcal{C}(z^\circ)$. Furthermore,

$$\lim_{\Omega^\circ_\pm\ni x\to z^\circ(\alpha)} v^\circ(x) = \mathscr{B}^\circ(\alpha)\mp\frac{1}{2}\frac{\varpi^\circ(\alpha)}{\partial_\alpha z^\circ(\alpha)^*},$$

where $\mathscr{B}^\circ\equiv\mathscr{B}(\omega^\circ)$ is

$$\begin{aligned}\mathscr{B}^\circ(\alpha)^* &:= \frac{1}{2\pi i}\mathrm{pv}\int_{\mathbb{T}}\frac{\varpi^\circ(\beta)}{z^\circ(\alpha)-z^\circ(\beta)}\,\mathrm{d}\beta\\ &= \frac{1}{2\pi i}\frac{\varpi^\circ(\alpha)}{\partial_\alpha z^\circ(\alpha)}\int_{\mathbb{T}}\frac{\partial_\alpha z^\circ(\alpha)-\partial_\alpha z^\circ(\beta)}{z^\circ(\alpha)-z^\circ(\beta)}\,\mathrm{d}\beta - \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{\varpi^\circ(\alpha)-\varpi^\circ(\beta)}{z^\circ(\alpha)-z^\circ(\beta)}\,\mathrm{d}\beta - \frac{1}{2}\frac{\varpi^\circ(\alpha)}{\partial_\alpha z^\circ(\alpha)}.\end{aligned} \tag{2.14}$$

Similarly, for $t > 0$ we deduce that $\bar{v}(t)$ is bounded in a neighborhood of $z_+(t) \cup z_-(t)$, and consequently $\bar{v}(t) = O((1+|x|)^{-1})$ in terms of $\|z(t)\|_{C^{1,\delta}}$, $\|\varpi(t)\|_{C^{0,\delta}}$, $\mathcal{A}(z(t))$ and $\mathcal{C}(z(t))$. Furthermore,

$$\lim_{\Omega^a_\pm(t)\ni x\to z_a(t,\alpha)} \bar{v}(x) = \mathscr{B}_a(t,\alpha) \mp \frac{1}{4}\frac{\varpi_a(t,\alpha)}{\partial_\alpha z_a(t,\alpha)^*},$$

where $\Omega^a_\pm(t) \equiv$ domain to the left/right side of z_a, and

$$\begin{aligned}
\mathscr{B}_a(t,\alpha)^* :=& \frac{1}{2}\sum_{b=\pm}\frac{1}{2\pi i}\mathrm{pv}\int_{\mathbb{T}}\frac{\varpi_b(t,\beta)}{z_a(t,\alpha)-z_b(t,\beta)}\,\mathrm{d}\beta \\
=& \frac{1}{2}\sum_{b=\pm}\left(\frac{1}{2\pi i}\frac{\varpi_b(t,\alpha)}{\partial_\alpha z_b(t,\alpha)}\int_{\mathbb{T}}\frac{\partial_\alpha z_b(t,\alpha)-\partial_\alpha z_b(t,\beta)}{z_a(t,\alpha)-z_b(t,\beta)}\,\mathrm{d}\beta\right. \\
&\left. -\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{\varpi_b(t,\alpha)-\varpi_b(t,\beta)}{z_a(t,\alpha)-z_b(t,\beta)}\,\mathrm{d}\beta - \theta_{a,b}\frac{\varpi_b(t,\alpha)}{\partial_\alpha z_b(t,\alpha)}\right),
\end{aligned} \tag{2.15}$$

where, for $a, b = \pm$, we have

$$\theta_{a,b} := \frac{1}{2\pi i}\mathrm{pv}\int_{z_b}\frac{\mathrm{d}x}{x-z_a} = \frac{1+\mathrm{sgn}(a-b)}{2}.$$

Let us focus on the integrands of the operator $\mathscr{B}_a$. The case $a = b$ can be seen as a Cauchy integral operator, which is well understood nowadays (see e.g. [50]). The case $a \neq b$ looks initially better because there is no singularity at $\alpha = \beta$ for $tc(\alpha) > 0$ (recall (2.7)). However, its derivatives get worse because, when they hit the denominator, the numerator does not compensate it, in contrast to the case $a = b$. This ends up requiring to control several integrals of the form

$$((\lambda-\mu)c(\alpha))^{k-j}\int_{\mathbb{T}}\frac{\beta^{j-1}\,\mathrm{d}\beta}{(z_\lambda(t,\alpha)-z_\mu(t,\alpha-\beta))^k},$$

in the regime $(\lambda-\mu)tc(\alpha) \neq 0$ and $j \leq k$.

2.2.1 Auxiliary lemmas

In this section we prove several lemmas which will be crucial to control the interaction between separate boundaries in chapters 4 and 6. They deal with several terms that appear recurrently when analyzing the Birkhoff-Rott and Muskat operators.

Lemma 2.2.1. *For all $k \in \mathbb{N}$, the function*

$$C^k_{\lambda,\mu}(t,\alpha) := \int_{\mathbb{T}}\frac{\beta^{k-1}}{(z_\lambda(t,\alpha)-z_\mu(t,\alpha-\beta))^k}\,\mathrm{d}\beta, \tag{2.16}$$

is uniformly bounded on $c(\alpha) > 0$, $0 < t \leq T$ and $\lambda, \mu \in [-1,1]$ with $\lambda \neq \mu$, in terms of $\mathcal{A}(z)$, $\mathcal{C}(z)$, $\|c\tau\|_{C^{1,\delta}}$ and $\|z\|_{C_tC^{1,\delta}}$.

Proof. First of all, by adding and subtracting $\partial_\alpha z'_\mu/\partial_\alpha z_\mu$ (recall sec. 1.3) we split

$$C^k_{\lambda,\mu}(t,\alpha) := \int_{\mathbb{T}}\frac{\beta^{k-1}}{(z_\lambda-z'_\mu)^k}\left(1-\frac{\partial_\alpha z'_\mu}{\partial_\alpha z_\mu}\right)\mathrm{d}\beta + \frac{1}{\partial_\alpha z_\mu}\int_{\mathbb{T}}\beta^{k-1}\frac{\partial_\alpha z'_\mu}{(z_\lambda-z'_\mu)^k}\,\mathrm{d}\beta. \tag{2.17}$$

The first term is controlled by $\mathcal{A}(z)$, $\mathcal{C}(z)$ and $\|\partial_\alpha z_\mu\|_{C_t C^{0,\delta}}$ (recall (2.7)). The identity (2.17) allows to prove the result by induction on k. For $k=1$, the second integral in (2.17) is explicit

$$\int_{\mathbb{T}} \frac{\partial_\alpha z'_\mu}{z_\lambda - z'_\mu}\,\mathrm{d}\beta = -2\pi i \theta_{\lambda,\mu},$$

where we have applied the Cauchy's argument principle for $\lambda \neq \mu$ and $tc(\alpha) > 0$. For $k \geq 2$, an integration by parts yields

$$\int_{\mathbb{T}} \beta^{k-1} \frac{\partial_\alpha z'_\mu}{(z_\lambda - z'_\mu)^k}\,\mathrm{d}\beta = -\frac{1}{k-1}\left(\frac{\beta}{z_\lambda - z'_\mu}\right)^{k-1}\Bigg|_{\beta=-\ell_\circ/2}^{\beta=+\ell_\circ/2} + \int_{\mathbb{T}} \frac{\beta^{k-2}}{(z_\lambda - z'_\mu)^{k-1}}\,\mathrm{d}\beta = S^{k-1}_{\lambda,\mu} + C^{k-1}_{\lambda,\mu},$$

where

$$S^j_{\lambda,\mu}(t,\alpha) := \frac{(-1)^j - 1}{j}\left(\frac{\ell_\circ/2}{z_\lambda(t,\alpha) - z_\mu(t,\alpha+\ell_\circ/2)}\right)^j. \tag{2.18}$$

The term $S^{k-1}_{\lambda,\mu}$ is controlled by $\mathcal{A}(z)$, $\mathcal{C}(z)$ while $C^{k-1}_{\lambda,\mu}$ is bounded by induction hypothesis. Furthermore, this recursive formula for $C^k_{\lambda,\mu}$ yields ($S^0_{\lambda,\mu} = 0$)

$$C^k_{\lambda,\mu} = \sum_{i=0}^{k-1} \frac{1}{(\partial_\alpha z_\mu)^{k-i}}\left(\int_{\mathbb{T}} \beta^i \frac{\partial_\alpha z_\mu - \partial_\alpha z'_\mu}{(z_\lambda - z'_\mu)^{i+1}}\,\mathrm{d}\beta + S^i_{\lambda,\mu}\right) - \frac{2\pi i \theta_{\lambda,\mu}}{(\partial_\alpha z_\mu)^k}, \tag{2.19}$$

which is controlled by $\mathcal{A}(z)$, $\mathcal{C}(z)$ and $\|\partial_\alpha z_\mu\|_{C_t C^{0,\delta}}$. ☐

Lemma 2.2.2. *For every $j, k \in \mathbb{N}$ with $j < k$ consider*

$$C^{j,k}_{\lambda,\mu}(t,\alpha) := ((\lambda-\mu)c(\alpha))^{k-j} \int_{\mathbb{T}} \frac{\beta^{j-1}\,\mathrm{d}\beta}{(z_\lambda(t,\alpha) - z_\mu(t,\alpha-\beta))^k}.$$

Then, $C^{j,k}_{\lambda,\mu}$ is uniformly bounded on $c(\alpha) > 0$, $0 < t \leq T$ and $\lambda, \mu \in [-1,1]$ with $\lambda \neq \mu$, in terms of $\mathcal{A}(z)$, $\mathcal{C}(z)$, $\|c\tau\|_{C^{k-j,\delta}}$, $\|z^{(n)}\|_{C^{k-j-n+1,\delta}}$ for $0 \leq n \leq k-j-1$ and $\|z^{(k-j)}\|_{C_t C^{1,\delta}}$. In particular, $(C^{j,k}_{\lambda,\mu})^{(0)}$ is uniformly bounded on $C^{l,\delta'}$ in terms of $\mathcal{C}(z^\circ)$ and $\|z^\circ\|_{C^{l+k-j+1,\delta}}$ for all $0 \leq \delta' < \delta$.

Proof. Let us denote

$$\Phi_{\lambda,\mu}(\alpha,\beta) := \frac{\beta}{z_\lambda(t,\alpha) - z_\mu(t,\alpha-\beta)},$$
$$\Psi_{\lambda,\mu}(\alpha,\beta) := \frac{t(\lambda-\mu)c(\alpha)}{z_\lambda(t,\alpha) - z_\mu(t,\alpha-\beta)}.$$

Both kernels are bounded by Lemma 2.1.1.

We will follow the double induction scheme on (j,k):

$$\begin{array}{lccccccccc}
k-j=0: & (1,1) & \to & (2,2) & \to & (3,3) & \to & \cdots & \to & (b,b) & \to & \cdots \\
 & & & \downarrow & & \downarrow & & & & \downarrow & & \\
k-j=1: & & & (1,2) & \to & (2,3) & \to & \cdots & \to & (b-1,b) & \to & \cdots \\
 & & & & & \downarrow & & & & \downarrow & & \\
k-j=2: & & & & & (1,3) & \to & \cdots & \to & (b-2,b) & \to & \cdots \\
 & & & & & & & & & & & \\
 & \cdots & & & & & & \cdots & & & & \cdots
\end{array}$$

where better regularity is required at every drop row. Notice that the first row correspond to Lemma 2.2.1.

We claim that the following identity holds

$$\begin{aligned}&((\lambda-\mu)c)^{j-k}C^{j,k}_{\lambda,\mu}\\&=\sum_{i=0}^{j-1}\frac{B^{j-1,k-1}_i}{(\partial_\alpha z_\mu)^{j-i}}\left(\int_{\mathbb{T}}\Delta_\beta\partial_\alpha z_\mu\Phi^{k-j+i+1}_{\lambda,\mu}\beta^{-(k-j)}\,\mathrm{d}\beta+S^{i,k-j+i}_{\lambda,\mu}\right)-\frac{2\pi i\theta_{\lambda,\mu}}{(\partial_\alpha z_\mu)^j}\delta_{j,k},\end{aligned}\tag{2.20}$$

where

$$B^{j,k}_i:=\binom{j}{i}\binom{k}{j-i}^{-1},\qquad S^{j,k}_{\lambda,\mu}(t,\alpha):=\frac{(-1)^j-1}{k}\frac{(\ell_\circ/2)^j}{(z_\lambda(t,\alpha)-z_\mu(t,\alpha+\ell_\circ/2))^k}.$$

We define the auxiliary functions

$$\tilde{C}^{j,k}_{\lambda,\mu}(t,\alpha):=\int_{\mathbb{T}}\beta^{j-1}\frac{\partial_\alpha z'_\mu}{(z_\lambda-z'_\mu)^k}\,\mathrm{d}\beta.$$

We can then write

$$((\lambda-\mu)c)^{j-k}C^{j,k}_{\lambda,\mu}=\frac{1}{\partial_\alpha z_\mu}\left(\int_{\mathbb{T}}\Delta_\beta\partial_\alpha z_\mu\Phi^k_{\lambda,\mu}\beta^{-(k-j)}\,\mathrm{d}\beta+\tilde{C}^{j,k}_{\lambda,\mu}\right).\tag{2.21}$$

In particular, the case $j=1$ in (2.20) follows from (2.21), because $B^{0,k-1}_0=1$, $S^{0,k}_{\lambda,\mu}=0$ and Cauchy's argument principle yields ($\lambda\neq\mu$)

$$\tilde{C}^{1,k}_{\lambda,\mu}=-2\pi i\theta_{\lambda,\mu}\delta_{1,k}.$$

For $j\geq 1$, an integration by parts yields

$$\begin{aligned}(j-1)((\lambda-\mu)c)^{j-k}C^{j-1,k-1}_{\lambda,\mu}&=\int_{\mathbb{T}}\frac{(j-1)\beta^{j-2}\,\mathrm{d}\beta}{(z_\lambda-z'_\mu)^{k-1}}\\&=\frac{\beta^{j-1}}{(z_\lambda-z'_\mu)^{k-1}}\Bigg|^{\beta=+\ell_\circ/2}_{\beta=-\ell_\circ/2}+(k-1)\int_{\mathbb{T}}\frac{\beta^{j-1}\partial_\alpha z_\mu(t,\alpha-\beta)}{(z_\lambda-z'_\mu)^k}\,\mathrm{d}\beta,\end{aligned}$$

that is,

$$(j-1)((\lambda-\mu)c)^{j-k}C^{j-1,k-1}_{\lambda,\mu}=(k-1)(\tilde{C}^{j,k}_{\lambda,\mu}-S^{j-1,k-1}_{\lambda,\mu}).$$

Hence, (2.21) reads as

$$\begin{aligned}&((\lambda-\mu)c)^{j-k}C^{j,k}_{\lambda,\mu}\\&=\frac{1}{\partial_\alpha z_\mu}\left(\int_{\mathbb{T}}\Delta_\beta\partial_\alpha z_\mu\Phi^k_{\lambda,\mu}\beta^{-(k-j)}\,\mathrm{d}\beta+S^{j-1,k-1}_{\lambda,\mu}+\frac{j-1}{k-1}((\lambda-\mu)c)^{j-k}C^{j-1,k-1}_{\lambda,\mu}\right),\end{aligned}$$

which allows to prove (2.20) by induction.

In light of the identity (2.20), to prove the result it is enough to control the integral term.

Let us assume w.l.o.g. that $i = j - 1$ for simplicity. Recalling section 1.3, by writing $z_\mu = z_\mu^{k-j-1)} + t^{k-j} z_\mu^{(k-j}$, we split it into

$$
\begin{aligned}
&((\lambda-\mu)c)^{k-j} \int_{\mathbb{T}} \Delta_\beta \partial_\alpha z_\mu \Phi^k_{\lambda,\mu} \beta^{-(k-j)}\, \mathrm{d}\beta \\
&= \sum_{n=0}^{k-j-1} \frac{((\lambda-\mu)c)^{k-j-n}}{n!} \int_{\mathbb{T}} \Delta_\beta \partial_\alpha z_\mu^{(n)} \Psi^n_{\lambda,\mu} \Phi^{k-n}_{\lambda,\mu} \beta^{n-(k-j)}\, \mathrm{d}\beta \\
&+ \int_{\mathbb{T}} \Delta_\beta \partial_\alpha z_\mu^{(k-j} \Psi^{k-j}_{\lambda,\mu} \Phi^j_{\lambda,\mu}\, \mathrm{d}\beta.
\end{aligned}
$$

On the one hand, the remainder term is controlled by $\|z_\mu^{(k-j}\|_{C^{1,\delta}}$. On the other hand, we split the other integrands into

$$
\begin{aligned}
&\int_{\mathbb{T}} \Delta_\beta \partial_\alpha z_\mu^{(n)} \Psi^n_{\lambda,\mu} \Phi^{k-n}_{\lambda,\mu} \beta^{n-(k-j)}\, \mathrm{d}\beta \\
&= (t(\lambda-\mu)c)^n \sum_{l=1}^{k-j-n} (-1)^{l-1} \frac{\partial_\alpha^{l+1} z_\mu^{(n)}}{l!} C^{j+l,k}_{\lambda,\mu} + \int_{\mathbb{T}} \Delta_\beta^{k-j-n+1} \partial_\alpha z_\mu^{(n)} \Psi^n_{\lambda,\mu} \Phi^{k-n}_{\lambda,\mu}\, \mathrm{d}\beta.
\end{aligned}
$$

The last term is bounded provided that $z_\mu^{(n)}$ belongs to $C^{k-j-n+1,\delta}$, and the terms $C^{j+l,k}_{\lambda,\mu}$ by induction hypothesis.

Finally, notice that the terms which do not vanish at $t = 0$ are those with $n = 0$ above. Moreover, for $t = 0$ these terms are Cauchy integral type operators. □

Chapter 3

Quantitative h-principle for a class of evolution equations

This chapter is based on the paper [26], joint work with Ángel Castro and Daniel Faraco.

3.1 Tartar framework

We consider the (inhomogeneous) first order linear system

$$\operatorname{div}_y(Lu) = f, \tag{3.1}$$

where $u : \mathbb{R}_+ \times \mathscr{D} \to \mathbb{R}^m$ is the unknown state variable, $f : \mathbb{R}_+ \times \mathscr{D} \to \mathbb{R}^m$ a force term and $\mathscr{D} \subset \mathbb{R}^n$ the open domain. We analyze the time and space variables $y = (t, x)$ separately. Thus, we split w.l.o.g. the linear map $L : \mathbb{R}^m \to \mathbb{R}^{m\times(1+n)}$ into

$$Lu = \left[\begin{array}{c|c} \pi(u) & Cu \\ \hline 0 & Su \end{array}\right],$$

where $\pi(u) := (u_1, \ldots, u_{m_0})$ represents the active variables, and the maps $C : \mathbb{R}^m \to \mathbb{R}^{m_0\times n}$ and $S : \mathbb{R}^m \to \mathbb{R}^{m_1\times n}$ $(m_0 + m_1 = m)$ correspond to the Cauchy and Stationary part of L respectively. As we are interested in evolution equations, we only consider the case $m_0 \geq 1$ (the stationary case, $m_0 = 0$ and $Lu = [0 \;\; Su]$, follows from [126, Appendix]). In case of $m_1 = 0$ we have $Lu = [u \;\; Cu]$.

The system (3.1) is coupled with the pointwise constraint

$$u(y) \in K_y, \quad y \in [0, T] \times \mathscr{D}, \tag{3.2}$$

where K is a closed subset of $[0, T] \times \mathscr{D} \times \mathbb{R}^m$ with $\emptyset \neq K_y := \{u : (y, u) \in K\}$ for all $y \in [0, T] \times \mathscr{D}$.

Both the initial datum u° and the force term f may be given or not. We only consider forces acting on the active variables, that is,

$$f = \left[\begin{array}{c} \pi(f) \\ \hline 0 \end{array}\right].$$

In case that the initial datum is given, we only need to specify the active part $\pi(u)^\circ$.

Examples

In the next chapters we will consider the following two examples in Hydrodynamics.

Example 3.1.1. As noticed in [54, 53], the incompressible Euler equation (1.9) can be written in the Tartar framework as follows. The state variable is $u = (v, M) \in \mathbb{R}^n \times \mathring{\mathbb{R}}^{n\times n}_{\mathrm{sym}} \simeq \mathbb{R}^{n+n(n+1)/2-1}$ where M encodes the traceless part of the non-linear term, $v\otimes v - \frac{1}{n}|v|^2 I_n$, and thus the pressure is corrected by $q = p + \frac{1}{n}|v|^2$. Then, the linear system is

$$Lu = \left[\begin{array}{c|c} v & M \\ \hline 0 & v \end{array}\right], \qquad f = -\left[\begin{array}{c} \nabla q \\ \hline 0 \end{array}\right].$$

For some fixed energy profile $e(y)$, the constraint is given in [54, 53] as

$$K_y = \{(v, M) \,:\, \tfrac{1}{2}|v|^2 = e(y),\, M = v\otimes v - \tfrac{1}{n}|v|^2 I_n\}.$$

Other related models have been brought to this framework: the compressible Euler equation (see e.g. [30, 29, 99, 68, 1, 98]), the inhomogeneous incompressible Euler equation ([76, 75]) and the ideal magnetohydrodynamics equation ([65, 9, 66]).

Example 3.1.2. The incompressible porous media equation (1.13)-(1.16), after rescaling in time (cf. chapter 5) can be written in the Tartar framework as follows. The state variable is $u = (\theta, v, m) \in \mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2 \simeq \mathbb{R}^5$ where m encodes the momentum ρv. Then, the linear system is

$$Lu = \left[\begin{array}{c|c} \theta & m \\ \hline 0 & v \\ 0 & i(v + Am + \theta i) \end{array}\right], \qquad f = 0,$$

where $A = \frac{\mu_h - \mu_l}{\mu_h + \mu_l}$ is the Atwood number. The constraint, which is y-independent, is

$$K = \{(\theta, v, m) \,:\, |\theta| = 1,\, m = \theta v\}.$$

Notice that the stationary part does not depend on m for $A = 0$. Furthermore, it reads as the velocity can be written as a Fourier multiplier $v = T(\theta)$ with symbol $\hat{T}(\xi) = -i\xi_1/\xi^*$. This case was first analyzed in [39] by Córdoba, Faraco and Gancedo. Remarkably, this was generalized by Shvydkoy ([122]) for a class of active scalar equations given by $v = T(\theta)$ whose symbol satisfies certain properties (see also [85]). However, the stationary part depends on m for $A \neq 0$, and thus this case was not included in the previous works. As we will see in chapter 5, this fact ends up hampering considerably the analysis of the relaxation.

3.2 Different concepts of solutions

In this section we recall the concepts of weak solution, relaxed solution and subsolution. From now on we fix an exponent $1 < p \leq \infty$. We will consider L^p endowed with the weak*-topology, that is $L^p_{w^*}$, which agrees with the weak-topology for $1 < p < \infty$.

Definition 3.2.1. We denote $L^p_S(\mathscr{D})$ by the closed subspace of $L^p_{w^*}(\mathscr{D};\mathbb{R}^m)$ of functions u satisfying

$$\mathrm{div}(Su) = 0,$$

in the sense of distributions, that is,

$$\int_{\mathscr{D}} Su\nabla\psi\,\mathrm{d}x = 0,$$

for all test function $\psi \in C_c^1(\mathscr{D})$.

Although we may impose additional linear properties on $L_S^p(\mathscr{D})$ (e.g. $\int u = 0$ for bounded domains, or non-flux boundary conditions) for simplicity we will keep here $L_S^p(\mathscr{D})$ as above and we will specify the particular cases where applicable.

Definition 3.2.2. We say that $u \in C([0,T]; L_S^p(\mathscr{D}))$ is a **weak solution** to (L) if it satisfies

$$\begin{aligned} \partial_t\pi(u) + \operatorname{div}(Cu) &= \pi(f), \\ \pi(u)|_{t=0} &= \pi(u)^\circ, \end{aligned}$$

in the sense of distributions, that is,

$$\int_0^t \int_{\mathscr{D}} (\pi(u)\partial_t\psi + Cu\nabla\psi)\,\mathrm{d}x\,\mathrm{d}s - \int_{\mathscr{D}} \pi(u)(t)\psi(t)\,\mathrm{d}x + \int_{\mathscr{D}} \pi(u)^\circ\psi^\circ\,\mathrm{d}x = -\langle\pi(f),\psi\rangle,$$

for all test function $\psi \in C_c^1(\mathbb{R}_+ \times \bar{\mathscr{D}})$ and $0 \leq t \leq T$. Moreover, we say that u is a **weak solution** to (L, K) if additionally

$$u(y) \in K_y, \quad y \in [0,T] \times \mathscr{D}. \tag{3.3}$$

Remark 3.2.1. Since these weak solutions u belong to the space $C_tL_{w^*}^p$, any pointwise property of u must be understood as

"almost everywhere in space at each time slice".

For instance, (3.3) reads as $u(t,x) \in K_{(t,x)}$ for almost every $x \in \mathscr{D}$ for all $0 \leq t \leq T$. We omit it for ease of the notation.

Other properties of u will be stated on some set $A \subset \mathscr{D}$ at some $t \in [0,T]$. In this case, we will simply denote

$$(t, A) \equiv \{t\} \times A.$$

Moreover, we will say that (t,A) is $\square$ if A is $\square$. For instance, we will say that (t,A) is a convex body if A is a convex body.

The pioneering work of Tartar ([129]) starts wondering about the nature of the weak*-limit of exact solutions to (L, K). These are called relaxed solutions.

Definition 3.2.3. We say that $u \in C([0,T]; L_S^p(\mathscr{D}))$ is a **relaxed solution** to (L, K) if there exists a sequence of weak solutions (u_k) to (L, K) satisfying $u_k \to u$ in $C_tL_S^p$.

These relaxed solutions are a recurrent theme in Hydrodynamics as they may contain macroscopic information usually studied in connection with hydrodynamical instabilities. However, in the problems we consider in this dissertation, it is really hard to construct such turbulent flows directly. In the groundbreaking work [54], De Lellis and Székelyhidi overcame this obstacle by taking first a candidate for the relaxed solution and then constructing the exact solutions by adding suitable correcting terms to it. More generally, this approach requires to derive first a candidate for the relaxed solution, called the subsolution in the convex integration framework, and then try to recover exact solutions from it. Let us give an heuristic argument which allows us to introduce several concepts from Lamination theory and the definition of subsolution.

Let $\mathscr{U}$ be the open subset of $(0,T]\times\mathscr{D}$ where we expect that the solutions behave wildly due to the singularities in u°. Roughly speaking, if an (hypothetical) solution u to (L,K) is averaged somehow, call the result $\bar{u}$, then $\bar{u}$ solves $(L,\bar{U})$ for some bigger constraint $\bar{U}$. As it is expected the fluctuation $u'=u-\bar{u}$ to be a highly oscillatory solution (in $\mathscr{U}$) to (L), it may look (locally) like a **plane wave**

$$\underline{u}h(k\xi\cdot y),$$

for some $\underline{u}\in\mathbb{R}^m$, $\xi\in\mathbb{R}\times\mathbb{S}^{n-1}$, $0\neq h\in C^1(\mathbb{T})$ with $\int h=0$ and $k\gg 1$. The set of directions $\underline{u}$ for which there is a plane wave solving (L) is called the Tartar **wave cone**

$$\Lambda := \{\underline{u}\in\mathbb{R}^m \,:\, (L\underline{u})\xi = 0 \;\text{ for some }\; \xi\in\mathbb{R}\times\mathbb{S}^{n-1}\}. \tag{3.4}$$

However, sometimes this cone contains directions which are not suitable to implement the convex integration method (see e.g. [66]). For such cases we need to restrict Λ to the good directions. In fact, notice that (3.4) excludes the directions which only produce oscillations in time $(\xi_0,0)$ (these were included in the original definition of the wave cone).

In this way, it seems that $\bar{U}$ may be given by the collection of Λ-laminations of K. The Λ-lamination of order 1 of K is defined as

$$K^{1,\Lambda} := \{\tfrac{1+s}{2}u_1 + \tfrac{1-s}{2}u_2 \,:\, s\in[-1,1],\, u_1,u_2\in K \text{ s.t. } u_1-u_2\in\Lambda\}, \tag{3.5}$$

and, inductively, the Λ-lamination of order $n\geq 2$ of K is defined as

$$K^{n,\Lambda} := (K^{n-1,\Lambda})^{1,\Lambda}.$$

This generates an ascending chain of sets $K\subset K^{1,\Lambda}\subset K^{2,\Lambda}\subset\cdots$ whose limit is called the **Λ-lamination hull** of K

$$K^{lc,\Lambda} := \bigcup_{n\geq 1} K^{n,\Lambda}.$$

As we mentioned, one would expect $K^{lc,\Lambda}$ to gather the states of relaxed solutions. However, there exist counterexamples (e.g. T_4-configurations) of constraints K without Λ-connected points, that is $K=K^{1,\Lambda}=K^{lc,\Lambda}$, but for which there are relaxed solutions taking values outside K (see e.g. [84]). There exist several notions of convex hulls that aims at fixing this drawback. One of them, which is commonly used in Hydrodynamics, is the **Λ-convex hull** of K, which is defined as follows. A state $u\in\mathbb{R}^m$ does not belong to K^Λ if there is a Λ-convex function f (meaning that $\lambda\mapsto f(u_0+\lambda\underline{u})$ is convex for all $u_0\in\mathbb{R}^m$ and $\underline{u}\in\Lambda$) so that $f\leq 0$ on K and $f(u)>0$. Of course it holds that $K^{lc,\Lambda}\subset K^\Lambda$.

Definition 3.2.4. We say that $\bar{u}\in C([0,T];L^p_S(\mathscr{D}))$ is a **subsolution** to (L,K) if it is a weak solution to (L) and

$$u(y)\in(K_y)^\Lambda, \quad y\in[0,T]\times\mathscr{D}.$$

Moreover, given $\emptyset\neq\mathscr{U}\subset(0,T]\times\mathscr{D}$ open, we say that $\bar{u}$ is a **strict** subsolution w.r.t. $\mathscr{U}$ if $\bar{u}$ is continuous on $\mathscr{U}$ and satisfies

$$u(y)\in\begin{cases} K_y, & y\notin\mathscr{U},\\ \operatorname{int}((K_y)^\Lambda), & y\in\mathscr{U}.\end{cases}$$

Therefore, one may set $\overline{U_y} = (K_y)^\Lambda$ and try to reach exact solutions from a subsolution. However, when the explicit computation of $(K_y)^\Lambda$ is unattainable due to the high complexity and dimensionality, it is more practical to consider a simpler but still large enough subset $\overline{U_y}$ of $(K_y)^{\mathrm{co}}$ (see [39, 122] and also [55, sec. 4]). When these correcting terms u' can be constructed and the set $\bar{U}$ satisfies some geometric and functional properties (see the next section) the convex integration method yields a homotopy principle whereby the problem of finding exact solutions is reduced to find a subsolution, a solution $\bar{u}$ to $(L, \bar{U})$. Schematically,

$$\begin{array}{ccccc} & (L,K) & \xrightarrow{\text{relaxation}} & (L,\bar{U}) & \\ & \big\downarrow & \text{h-principle} & \big\downarrow & \\ \text{solution} & u & \xleftarrow[\text{integration}]{\text{convex}} & \bar{u} & \text{subsolution} \end{array} \tag{3.6}$$

3.3 H-principle

3.3.1 Hypothesis

In the setting of the h-principle we will assume that the following three hypothesis hold ([53, 126, 26]). The first one (H1) provides localized plane wave solutions to (L) for some cone Λ. The second hypothesis (H2) provides long Λ-segments inside some open sets U_y with $K_y \subset \partial U_y$ for all $y \in \mathscr{U}$. The last one (H3) controls the $C_tL^p_S$-limit of strict subsolutions.

Although these results admit other versions by changing (H1)-(H3), we will prove the version that we need and we will comment possible generalizations where appropriate.

(H1) Localized plane waves. There is a cone $\Lambda \subset \mathbb{R}^m$ and a profile $0 \neq h \in C^1(\mathbb{T}; [-1,1])$ with $\int h = 0$ satisfying: For all $\underline{u} \in \Lambda$ and $\psi \in C_c^\infty(\mathbb{R}^{n+1})$ there is $\xi \in \mathbb{R} \times \mathbb{S}^{n-1}$ for which there are smooth solutions to (L) of the form

$$u'_k(y) = \underline{u} h(k\xi \cdot y)\psi(y) + O(k^{-1}),$$

where O only depends on $|\underline{u}|$, $|\xi|$ and $\{|D^\alpha \psi(y)| : 1 \leq |\alpha| \leq N\}$ for some fixed N.

(H2) Long Λ-segments. There are open sets $\emptyset \neq \mathscr{U} \subset [0,T] \times \mathscr{D}$ and $U \subset \mathscr{U} \times \mathbb{R}^m$ with $\emptyset \neq U_y \subset (K_y)^{\mathrm{co}} \setminus K_y$ for all $y \in \mathscr{U}$, satisfying:

1. There is $a \in C(\mathscr{U}; \mathbb{R}_+)$ satisfying: For all $u \in \overline{U_y}$, $|\pi(u)| = a(y)$ iff $u \in K_y$.

2. There is an increasing $\phi \in C((0,\infty); (0,1])$ satisfying: For all $(y,u) \in U$ there is $\underline{u} \in \Lambda$ with $|\pi(\underline{u})|^2 \geq \phi(a(y)^2 - |\pi(u)|^2)$ such that

$$u + [-\underline{u}, \underline{u}] \subset U_y.$$

(H3) $C_tL^p_S$-compactness. Let us denote X_0 by the subset of $C([0,T]; L^p_S(\mathscr{D}))$ formed by weak solutions u to (L) with u continuous on $\mathscr{U}$ and satisfying

$$u(y) \in \begin{cases} K_y, & y \notin \mathscr{U}, \\ U_y, & y \in \mathscr{U}. \end{cases}$$

We assume that there exists $\bar{u} \in X_0$ for which the space

$$X_0(\bar{u}) := \{u \in X_0 \,:\, u(y) = \bar{u}(y),\ y \notin \mathscr{U}\},$$

satisfies:

1. $X_0(\bar{u})$ is bounded in $C_t L^p$.
2. For all $X_0(\bar{u}) \ni u_k \to u$ in $C_t L^p_{w^*}$, it holds that

$$u(y) \in \overline{U_y}, \quad y \in \mathscr{U}.$$

Definition 3.3.1. By (H3) there is a compact and metrizable subset B of $L^p_S(\mathscr{D})$ satisfying $X_0(\bar{u}) \subset C([0,T];B)$. The metric d_B induces naturally a metric on $C([0,T];B)$

$$\sup_{0\le t\le T} d_B(u(t), v(t)), \qquad u, v \in C([0,T];B),$$

which agrees with the natural topology of $C([0,T];B)$ as a subset of $C_t L^p_{w^*}$. We define $X(\bar{u})$ as the closure of $X_0(\bar{u})$ in $C([0,T];B)$. As a result, $X(\bar{u})$ becomes a complete metric subspace of $C([0,T];B)$.

Recall that in a metric space a set is **nowhere dense** if its closure has empty interior. A **residual** set is then the complement of a countable union of nowhere dense sets. By the Baire category theorem, a residual set is dense. With these preparations, the h-principle read as follows.

Theorem 3.3.1 (H-principle)**.** *Assuming (H1)-(H3), the set*

$$\{u \in X(\bar{u}) \,:\, u(y) \in K_y,\ y \in \mathscr{U}\}$$

contains a residual set R in $X(\bar{u})$. Hence, there are infinitely many weak solutions to (L, K).

3.3.2 Microscopic behavior

In this section we investigate how wild the solutions from Theorem 3.3.1 can behave inside $\mathscr{U}$. This will be characterized in terms of the fractional Sobolev spaces.

Definition 3.3.2. Given $\Omega \subset \mathbb{R}^n$ open, $0 < s < 1$ and $1 \le q < \infty$, we consider the fractional Sobolev space

$$W^{s,q}(\Omega) := \{f \in L^q(\Omega) \,:\, |f|_{W^{s,q}(\Omega)} < \infty\},$$

characterized by the Gagliardo semi-norm

$$|f|_{W^{s,q}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^q}{|x-y|^{n+sq}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{q}}.$$

This forms a descending chain of (continuously embedded) sets: For $0 < s_0 < s_1 < 1$

$$W^{s_1,q}(\Omega) \hookrightarrow W^{s_0,q}(\Omega).$$

The open set Ω is called an extension domain if it admits an extension operator $E : W^{s,q}(\Omega) \hookrightarrow W^{s,q}(\mathbb{R}^n)$. If this case (e.g. $\partial\Omega$ of class $C^{0,1}$) it holds that (see e.g. [58])

$$W^{1,q}(\Omega) \hookrightarrow W^{s,q}(\Omega) \hookrightarrow L^q(\Omega)$$

and moreover

$$C_1\|Ef\|_{W^{1,q}(\mathbb{R}^n)} \underset{s\uparrow 1}{\longleftarrow} s(1-s)|Ef|_{W^{s,q}(\mathbb{R}^n)} \underset{s\downarrow 0}{\longrightarrow} C_0\|Ef\|_{L^q(\mathbb{R}^n)}$$

for some constants $0 < C_0, C_1 < \infty$. In other words, the spaces $W^{s,q}$ form a continuous interpolation between $W^{1,q}$ and L^q, where less regularity is required as $s \to 0$.

With these preparations, the microscopic part of the h-principle reads as follows.

Theorem 3.3.2 (Microscopic h-principle). *Assuming (H1)-(H3) and that $a \in C_tC^1$, there is a residual set R_{wild} in $X(\bar{u})$ formed by weak solutions u to (L,K) satisfying*

$$\pi(u(t)) \notin W^{s,q}(\Omega), \tag{3.7}$$

for every open $(t,\Omega) \subset \mathscr{U}$, $0 < s < 1$ and $1 \leq q < \infty$.

It is easy to check that, for all $s < \delta$,

$$C^{\delta}(\Omega) \hookrightarrow W^{s,q}(\Omega).$$

As a corollary, the above h-principle shows the existence of weak solutions u such that $\pi(u)$ is almost nowhere Hölder continuous inside $\mathscr{U}$ (cf. Remark 1.2.1).

3.3.3 Macroscopic behavior

In this section we investigate how close in average the weak solutions u from Theorem 3.3.1 can be to the subsolution $\bar{u}$ inside $\mathscr{U}$. Thus, we would like to estimate

$$\fint_C (u-\bar{u})(t,x)\,\mathrm{d}x \tag{3.8}$$

for some sets $(t,C) \subset \mathscr{U}$. What can we expect? As we mentioned in section 1.2, the Lebesgue differentiation theorem says that

$$\lim_{\substack{C\to\{x_0\}\\ \text{regular}}} \fint_C u(t,x)\,\mathrm{d}x = u(t,x_0) \in K_{(t,x_0)},$$

where u may take values in K wildly because of Theorem 3.3.2. This avoids to estimate u from $\bar{u}$ for arbitrarily small sets C. However, there is still room to control (3.8) in many situations.

Once we know how to control (3.8), it is not difficult to extend the result for more quantities, which will be useful in the next chapters. We will obtain the upper bound

$$\left|\fint_{Z(t,C)} (F(u)-F(\bar{u}))(t,x)g(t,x)\,\mathrm{d}x\right| \leq E(t,C),$$

on convex bodies (t,C) (recall Rem. 3.2.1), for some weak*-continuous functional F, suitable weight g, parametrization Z of $\mathscr{U}$ and error E in terms of the size of C and the distance to the (space-time) boundary of $\mathscr{U}$.

Definition 3.3.3. We fix a finite family $\mathcal{F}$ of 4-tuples (F, g, Z, E) satisfying:

- There are $1 \leq p_1, p_2, p_3 \leq \infty$ with $p_3 \neq \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$ so that $F : \mathbb{R}^m \to \mathbb{R}$, $g : \mathscr{U} \to \mathbb{R}$ with $F \in C(L^p_S; L^{p_1}_{w^*})$ and $g \in C_t L^{p_2}_{\text{loc}}$.
- There is an open set $\mathscr{U}' \subset [0, T] \times \mathbb{R}^n$ so that the map $\mathbf{Z} : \mathscr{U}' \to \mathscr{U}$ given by $\mathbf{Z}(t, x) := (t, Z(t, x))$ is a $C_t C^1_x$-homeomorphism.
- There are $0 \leq \gamma < \frac{1}{p_3}$ and $D \in C(\mathscr{U}'; (0, 1])$ such that
$$E(t, C) := \frac{1 \wedge |Z(t, C)|^\gamma}{|Z(t, C)|} \sup_{x \in C} D(t, x),$$
on convex bodies $(t, C) \subset \mathscr{U}'$, where $a \wedge b \equiv \min\{a, b\}$.

Remark 3.3.1. The functional F has been introduced to consider more general weak*-continuous quantities (not necessarily linear, see [26]). Then, the weight g appears naturally and it has proven to be useful in some situations. The map Z plays the role of the parametrization for the turbulence zone given in (2.3). The term D defining E has been introduced to show that the error depends on the distance to the (space-time) boundary of $\mathscr{U}$, and the parameter γ to refine this estimate for small C's. However, for simplicity one may consider $F(u) = u$, $g = 1$, $Z(t, x) = x$, $\gamma = 0$ and $D = \text{dist}_{\partial \mathscr{U}}$, for which
$$E(t, C) = \frac{1}{|C|} \sup_{x \in C} \text{dist}_{\partial \mathscr{U}}(t, x).$$
This contains relevant information and it is easier to understand in a first reading.

Definition 3.3.4. We define $X_0(\bar{u}, \mathcal{F})$ as the subset of $X_0(\bar{u})$ formed by functions u for which there is a constant $0 < c(u) < 1$ satisfying
$$\left| \fint_{Z(t,C)} (F(u) - F(\bar{u}))(t, x) g(t, x) \, \mathrm{d}x \right| \leq c E(t, C),$$
for every convex body $(t, C) \subset \mathscr{U}'$ and $(F, g, Z, E) \in \mathcal{F}$. Analogously to $X(\bar{u})$, we define $X(\bar{u}, \mathcal{F})$ as the closure of $X_0(\bar{u}, \mathcal{F})$, which becomes a complete metric subspace of $X(\bar{u})$.

Lemma 3.3.1. *Any $u \in X(\bar{u}, \mathcal{F})$ satisfies*
$$\left| \fint_{Z(t,C)} (F(u) - F(\bar{u}))(t, x) g(t, x) \, \mathrm{d}x \right| \leq E(t, C), \tag{3.9}$$
for every convex body $(t, C) \subset \mathscr{U}'$ and $(F, g, Z, E) \in \mathcal{F}$.

Proof. Take $(u_k) \subset X_0(\bar{u}, \mathcal{F})$ converging to u. Fix a convex body $(t, C) \subset \mathscr{U}'$ and $(F, g, Z, E) \in \mathcal{F}$. By adding and subtracting $F(u_k)$ and applying Definition 3.3.4, we get
$$\begin{aligned} &\left| \fint_{Z(t,C)} (F(u) - F(\bar{u}))(t, x) g(t, x) \, \mathrm{d}x \right| \\ \leq &\left| \fint_{Z(t,C)} (F(u) - F(u_k))(t, x) g(t, x) \, \mathrm{d}x \right| + c(u_k) E(t, C), \end{aligned}$$
with $0 < c(u_k) < 1$. Since $F(u_k(t)) \to F(u(t))$ in $L^{p_1}_{w^*}$ and $g(t) \mathbb{1}_{Z(t,C)} \in L^{p_2}$, the first term vanishes as $k \to \infty$. This proves (3.9). $\square$

With these preparations, the macroscopic part of the h-principle reads as follows.

Theorem 3.3.3 (Macroscopic h-principle)**.** *Assuming (H1)-(H3), the set*

$$\{u \in X(\bar{u}, \mathcal{F}) \,:\, u(y) \in K_y,\ y \in \mathscr{U}\},$$

contains a residual set $R_{\mathcal{F}}$ in $X(\bar{u}, \mathcal{F})$. Hence, there are infinitely many weak solutions to (L, K) satisfying (3.9)*.*

Observe that Theorem 3.3.3 would follow from Theorem 3.3.1 and Lemma 3.3.1 if $X_0(\bar{u}, \mathcal{F})$ was open because, in such a case, we might take simply $R_{\mathcal{F}} = R \cap X_0(\bar{u}, \mathcal{F})$. However, this seems to be false since it requires a stronger convergence than d_X. To overcome this drawback we adapt the proof of Theorem 3.3.1 to our new space $X(\bar{u}, \mathcal{F})$.

3.4 Convex integration scheme

In this section we prove the Theorems 3.3.1-3.3.3. First of all let us recall several notions in Baire category theory ([117]).

A function $J : X \to \mathbb{R}$ is **Baire-1** if it is pointwise limit of continuous functions, e.g. if J is upper-semicontinuous ($\limsup_{u \to u_0} J(u) \le J(u_0)$ for all $u_0 \in X$). The set of continuity points of a Baire-1 function X

$$X_J := \{u \in X \,:\, J \text{ is continuous at } u\},$$

is a countable intersection of open and dense subsets of X, thus residual.

From now on we consider the complete metric space $X \equiv X(\bar{u}, \mathcal{F})$. We would like to measure the distance of functions in X from being exact solutions to (L, K). Let us introduce the distance

$$d(y, u) := e(y) - c(u),$$

where $e(y) := a(y)^2$ and $c(u) := |\pi(u)|^2$ (recall (H2.1)). More generally, we may consider in (H2.1) any convex function satisfying: $c(u) = e(y)$ iff $u \in K_y$, for all $u \in \overline{U_y}$ and $y \in \mathscr{U}$. This would allow to consider $c(u) = |\pi(u)|_p^p = \sum_{j=1}^{m_0} |u_j|^p$ in order to control other energy profiles. We notice that $c(u) = |\pi(u)|^2$ satisfies

$$|\pi(u+v)|^2 = |\pi(u)|^2 + 2\pi(u) \cdot \pi(v) + |\pi(v)|^2, \qquad u, v \in \mathbb{R}^m.$$

More generally, we may assume that there is $G \in C(\mathbb{R}^m; \mathbb{R}^m)$ and $0 \neq H \in C(\mathbb{R}^m)$ positive homogeneous of degree $r \geq 1$ such that

$$c(u+v) \geq c(u) + G(u) \cdot v + H(v), \qquad u, v \in \mathbb{R}^m. \tag{3.10}$$

This condition can be understood as a weakening of the classical strongly convexity, for which we recall $G = \nabla c$ and $H(v) \propto |v|^2$. On the one hand, this notion admits directions where H can be zero. On the other hand, it does not require the Hessian to be uniformly definite positive.

For any (space-time) cylinder $C = I \times Q \subset\subset \mathscr{U}$ with side length ℓ, where I denotes the time interval and Q the space cube, we consider the functional on C

$$\begin{aligned} J : X &\to \mathbb{R}_+ \\ u &\mapsto \sup_{t \in I} \int_Q d(t, x, u(t,x))\, \mathrm{d}x. \end{aligned}$$

Notice that any $u \in X$ satisfies $u(y) \in (\overline{U_y})^{\mathrm{co}}$ by the Mazur's lemma. Since $\overline{U_y} \subset (K_y)^{\mathrm{co}}$ and $d(y,\cdot)$ is concave and vanishes at K_y, then $d(y,u) \geq 0$ for all $u \in (K_y)^{\mathrm{co}}$. Moreover, J is bounded from above. Hence, J is well defined.

In the next lemma we show that X_J is residual (see [53, Lemma 4]).

Lemma 3.4.1. *J is upper-semicontinuous. Hence, X_J is countable intersection of open and dense subsets of X.*

Proof. We prove it by contradicition. Let us suppose that there exists a sequence (u_k) and u in X with $u_k \to u$ but satisfying

$$\underbrace{\sup_{t\in I}\int_Q d(t,x,u(t,x))\,\mathrm{d}x}_{J(u)} < \limsup_{k\to\infty} \underbrace{\sup_{t\in I}\int_Q d(t,x,u_k(t,x))\,\mathrm{d}x}_{J(u_k)}. \tag{3.11}$$

For any $k \in \mathbb{N}$ there is $t_k \in I$ satisfying

$$\sup_{t\in I}\int_Q d(t,x,u_k(t,x))\,\mathrm{d}x < \int_Q d(t_k,x,u_k(t_k,x))\,\mathrm{d}x + 2^{-k}.$$

By applying the lim sup above, (3.11) implies

$$\sup_{t\in I}\int_Q d(t,x,u(t,x))\,\mathrm{d}x < \limsup_{k\to\infty}\int_Q d(t_k,x,u_k(t_k,x))\,\mathrm{d}x. \tag{3.12}$$

Since I is compact, we may assume (for a subsequence if necessary) that $t_k \to t_0 \in I$. Finally (recall Def. 3.3.1) notice that $u_k \to u$ in C_tB implies

$$\begin{aligned} d_B(u_k(t_k),u(t_0)) &\leq d_B(u_k(t_k),u(t_k)) + d_B(u(t_k),u(t_0)) \\ &\leq d_X(u_k,u) + d_B(u(t_k),u(t_0)) \to 0, \end{aligned}$$

that is, $u_k(t_k) \to u(t_0)$ in $L^p_{w^*}$. Hence, using (3.10), we get

$$\int_Q c(u_k(t_k))\,\mathrm{d}x \geq \int_Q c(u(t_0))\,\mathrm{d}x + \int_Q G(u(t_0))\cdot(u_k(t_k)-u(t_0))\,\mathrm{d}x + \int_Q H(u_k(t_k)-u(t_0))\,\mathrm{d}x.$$

Notice that the third term is positive and the second one vanishes as $k \to 0$. Therefore,

$$\liminf_{k\to\infty}\int_Q c(u_k(t_k))\,\mathrm{d}x \geq \int_Q c(u(t_0))\,\mathrm{d}x. \tag{3.13}$$

Finally, using that $e(t_k) \to e(t_0)$ and (3.13), we deduce that

$$\limsup_{k\to\infty}\int_Q d(t_k,x,u_k(t_k,x))\,\mathrm{d}x \leq \int_Q d(t_0,x,u(t_0,x))\,\mathrm{d}x,$$

which contradicts (3.12). □

The following lemma is nothing but a simple observation in Young measure theory (see [53, Lemma 7]). This can be understood as a generalization of the Riemann-Lebesgue Lemma. For our purpose, since the convex integration method is based on adding the perturbations $u'_k = \underline{u}h\psi + O(k^{-1})$ from (H1) to a given $u \in X_0$, for $f = \underline{u}h$ and $A = \mathrm{id}$, this lemma will imply that $u + u'_k \to u$, whereas for $A = H$ as in (3.10) it will imply that $J(u+u'_k) \nrightarrow J(u)$.

Lemma 3.4.2. *Let $f \in L^\infty(\mathbb{T};\mathbb{R}^m)$ and $\xi \in \mathbb{R}\times\mathbb{S}^{n-1}$. Then, for any open $\Omega \subset \mathbb{R}^n$, $g \in L^1(\Omega)$ and $A \in C(\mathbb{R}^m)$, we have*

$$\int_\Omega g(x)A(f(k\xi\cdot(t,x)))\,\mathrm{d}x \to \int_\Omega g(x)\,\mathrm{d}x \int_{\mathbb{T}} A(f(\tau))\,\mathrm{d}\tau, \tag{3.14}$$

uniformly in $t \in \mathbb{R}$ as $k \to \infty$.

Proof. First assume that $g \in C_c^\infty(\Omega)$. If $\xi = (\xi_0,\zeta)$, take an orthonormal basis $\{\zeta_i\}$ of $\mathbb{R}^n$ with $\zeta_1 = \zeta$ and $O = [\zeta_1|\cdots|\zeta_n] \in \mathrm{SO}(n)$. We make first the change of variables $x = Ox' = \sum_{i=1}^n x_i'\zeta_i$ with $\Omega' = O^\dagger\Omega$ and $G(x') := g(Ox')$

$$\int_\Omega g(x)A(f(k\xi\cdot(t,x)))\,\mathrm{d}x = \int_{\Omega'} G(x')A(f(k\xi_0 t + kx_1'))\,\mathrm{d}x',$$
$$\int_\Omega g(x)\,\mathrm{d}x = \int_{\Omega'} G(x')\,\mathrm{d}x'.$$

After that, we integrate by parts

$$\int_{\Omega'} G(x')A(f(k\xi_0 t + kx_1'))\,\mathrm{d}x' = -\frac{1}{k}\int_{\Omega'} \partial_1 G(x')\int_0^{kx_1'} A(f(k\xi_0 t+\tau))\,\mathrm{d}\tau\,\mathrm{d}x',$$
$$\int_{\Omega'} G(x')\,\mathrm{d}x' = -\int_{\Omega'} \partial_1 G(x')x_1'\,\mathrm{d}x'.$$

Finally, by adding and subtracting the term

$$\frac{1}{k}\int_{\Omega'} \partial_1 G(x')\int_0^{\lceil kx_1'\rceil} A(f(k\xi_0 t+\tau))\,\mathrm{d}\tau\,\mathrm{d}x' = \frac{1}{k}\int_{\Omega'} \partial_1 G(x')\lceil kx_1'\rceil\int_0^1 A(f(\tau))\,\mathrm{d}\tau\,\mathrm{d}x',$$

where $\lceil\cdot\rceil$ is the ceiling function, we get

$$\begin{aligned}
&\left|\int_\Omega g(x)A(f(k\xi\cdot(t,x)))\,\mathrm{d}x - \int_\Omega g(x)\,\mathrm{d}x\int_0^1 A(f(\tau))\,\mathrm{d}\tau\right| \\
&= \frac{1}{k}\left|\int_{\Omega'} \partial_1 G(x')\left[\int_0^{\lceil kx_1'\rceil - kx_1'} A(f(k\xi_0 t+\tau))\,\mathrm{d}\tau + (kx_1' - \lceil kx_1'\rceil)\int_0^1 A(f(\tau))\,\mathrm{d}\tau\right]\,\mathrm{d}x'\right| \\
&\leq \frac{2}{k}\|\partial_1 G\|_{L^1(\Omega')}\|A\|_{C^0(B_f)},
\end{aligned}$$

being B_f the ball of radius $\|f\|_{L^\infty(\mathbb{T})}$. Therefore, (3.14) follows. By density, the result is extended for all $g \in L^1(\Omega)$. □

The following lemma shows that the Λ-segments in (H2) can be selected uniformly away from the boundary on compact sets.

Lemma 3.4.3. *For every $(y,u) \in U$ we consider*

$$\Lambda_{(y,u)} := \{\underline{u} \in \Lambda : H(\underline{u}) \geq \phi(d(y,u)),\ u + [-\underline{u},\underline{u}] \subset U_y\},$$

which is non-empty by (H2). Then, the function

$$(y,u) \mapsto \sup_{\underline{u}\in\Lambda_{(y,u)}} \mathrm{dist}((y,u+[-\underline{u},\underline{u}]),\partial U) \tag{3.15}$$

is lower-semicontinuous on U. Hence, given $V \subset\subset U$, there are $\delta, \varepsilon > 0$ such that, for all $(y_0, u_0) \in V$ there is $\underline{u}_0 \in \Lambda_{(y_0,u_0)}$ satisfying

$$\mathrm{dist}(u + [-\underline{u}_0, \underline{u}_0], \partial U_y) \geq \delta,$$

for all $(y, u) \in V$ with $|(y, u) - (y_0, u_0)| \leq \varepsilon$.

Proof. Let us denote $f : U \to \mathbb{R}_+$ by the function given in (3.15). Fix $(y, u) \in U$. For any $(y_k, u_k) \subset U$ converging to (y, u) we consider

$$\delta_k \equiv \left| \left(\frac{\phi(d(y_k, u_k))}{\phi(d(y, u))} \right)^{\frac{1}{r}} - 1 \right|, \tag{3.16}$$

with r the degree of homogeneity of H. Since $\phi \circ d$ is continuous and positive on U, we have $\delta_k \to 0$ as $k \to \infty$. By definition, for any $0 < \varepsilon < \frac{1}{2}$ there is $\underline{u} \in \Lambda_{(y,u)}$ satisfying

$$\mathrm{dist}((y, u + [-\underline{u}, \underline{u}]), \partial U) \geq (1 - \varepsilon) f(y, u) > 0.$$

Let us take $\underline{u}_k \equiv \lambda_k \underline{u}$ with $\lambda_k \equiv 1 + \delta_k$. We claim that there is a big enough k_0 so that $\underline{u}_k \in \Lambda_{(y_k,u_k)}$ for all $k \geq k_0$. On the one hand, by using the r-homogeneity of H, that $\underline{u} \in \Lambda_{(y,u)}$ and (3.16), we get

$$H(\underline{u}_k) = \lambda_k^r H(\underline{u}) \geq \lambda_k^r \phi(d(y, u)) \geq \phi(d(y_k, u_k)).$$

On the other hand, by adding and subtracting $(y, u + \lambda\underline{u})$, the triangle inequality implies

$$\begin{aligned} \mathrm{dist}((y_k, u_k + [-\underline{u}_k, \underline{u}_k]), \partial U) &= \min_{\substack{|\lambda| \leq 1 \\ (y', u') \in \partial U}} |(y_k, u_k + \lambda\lambda_k \underline{u}) - (y', u')| \\ &\geq \mathrm{dist}((y, u + [-\underline{u}, \underline{u}]), \partial U) - |(y_k, u_k) - (y, u)| - \delta_k |\underline{u}|. \end{aligned}$$

Hence, for a big enough k_0 we have $\mathrm{dist}((y_k, u_k + [-\underline{u}_k, \underline{u}_k]), \partial U) \geq (1 - 2\varepsilon) f(y, u)$ and so $u_k + [-\underline{u}_k, \underline{u}_k] \subset U_{y_k}$ for all $k \geq k_0$. Finally, since for all $k \geq k_0$ we have

$$f(y_k, u_k) \geq \mathrm{dist}((y_k, u_k + [-\underline{u}_k, \underline{u}_k]), \partial U) \geq (1 - 2\varepsilon) f(y, u),$$

by applying first the lim inf above and then making $\varepsilon \to 0$ we conclude that f is lower-semicontinuous at (y, u).

Given $V \subset\subset U$, let $(y_k, u_k) \subset \bar{V}$ be a minimizing sequence of f. Since $\bar{V}$ is compact, we may assume (for a subsequence if necessary) that $(y_k, u_k) \to (y_0, u_0) \in \bar{V}$. Then, the lower-semicontinuity of f implies

$$\inf_{(y,u) \in \bar{V}} f(y, u) = \lim_{k \to \infty} f(y_k, u_k) \geq f(y_0, u_0) > 0.$$

Hence, for all $(y, u) \in V$ there is $\underline{u} \in \Lambda_{(y,u)}$ such that

$$W \equiv \bigcup_{(y,u) \in V} (y, u + [-\underline{u}, \underline{u}]) \subset\subset U,$$

and so $B_\varepsilon(W) \subset\subset U$ for some $\varepsilon > 0$. $\square$

The key point to prove the h-principle is the following perturbation property. The steps 1, 2 and 4 in the proof are an adaptation of the proof of [53, Prop. 3]. They allow to prove prove Theorem 3.3.1. We recall them for convenience. The step 3 is the new requirement from $X_0(\bar{u}, \mathcal{F})$ and our main contribution to this scheme. More precisely, although the approximating sequence is constructed in the same way as in [53], which belongs to $X_0(\bar{u})$, we need to check that it belongs to our $X_0(\bar{u}, \mathcal{F})$. This allow to prove Theorem 3.3.3. We present the four steps together as Theorem 3.3.1 can be deduced from Theorem 3.3.3 for $X(\bar{u}, \emptyset) = X(\bar{u})$.

Proposition 3.4.1 (Perturbation property)**.** *For all $\alpha > 0$ there exists $\beta(\alpha, C) > 0$ such that, whenever $u \in X_0(\bar{u}, \mathcal{F})$ satisfies*

$$J(u) \geq \alpha,$$

there exists a sequence $(u_k) \subset X_0(\bar{u}, \mathcal{F})$ with $u_k \to u$ and

$$\limsup_{k\to\infty} J(u_k) \leq J(u) - \beta. \tag{3.17}$$

Proof. The proof is split in 4 steps. In the *step 1* we recall how

$$I_0 \ni t \mapsto \int_{Q_0} d(t, x, u(t, x))\,\mathrm{d}x,$$

is discretized. In the *step 2* we construct the sequence (u_k) converging to u. In the *step 3* we check that $(u_k) \subset X_0(\bar{u}, \mathcal{F})$. In the *step 4* we prove (3.17).

Step 1. The discretization. Let us split the (space-time) cylinder $C_0 = I_0 \times Q_0$

$$C_0 = y_0 + \ell_0[0, 1]^{n+1},$$

in a grid of cubes of size $\varepsilon = 2^{-N}\ell_0 > 0$, for some $N \in \mathbb{N}$ to be determined. In order to perturb u at ∂I_0, it is convenient to augment I_0 as $I_\varepsilon \equiv \bar{B}_\varepsilon(I_0) \cap [0, T]$. In particular, we take $0 < \varepsilon \leq \varepsilon_1 = 2^{-N_1}\ell_0 \leq \frac{t_0}{2}$ satisfying $I_{\varepsilon_1} \times Q_0 \subset\subset \mathscr{U}$.

1.1. The shifted grid. For any $(i, j) \in \mathbb{N}_0 \times \mathbb{N}_0^n$ with $\max(j_1, \ldots, j_n) \leq 2^N - 1$ and $i \leq 2^N + p(j) - 1$, where $p \in \{0, 1\}$ is the parity of $|j| \equiv j_1 + \ldots + j_n \in 2\mathbb{Z} + p$, we define the (space-time) cylinders (see Figure 3.1)

$$C_\varepsilon^{i,j} \equiv y_0 + \varepsilon(i - \tfrac{p}{2}, j) + \varepsilon[0, 1]^{n+1}.$$

Notice that these cylinders cover C_0. The cylinders with $p(j) = 1$ have been shifted in time to perturb the solution at each time slice (see Figure 3.1). For each $p \in \{0, 1\}$, we define

$$\mathcal{C}_\varepsilon^p \equiv \{C_\varepsilon^{i,j} : |j| \in 2\mathbb{Z} + p\},$$

and also $\mathcal{C}_\varepsilon \equiv \mathcal{C}_\varepsilon^0 \cup \mathcal{C}_\varepsilon^1$. For each cylinder $C \in \mathcal{C}_\varepsilon$ we denote y_C by its center and we consider its reduced version

$$C_{\mathrm{red}} \equiv y_C + \tfrac{3}{4}\varepsilon[-\tfrac{1}{2}, \tfrac{1}{2}]^{n+1}.$$

We split C as $C = I \times Q$ with I the time interval and Q the space cube, and similarly $C_{\mathrm{red}} = I_{\mathrm{red}} \times Q_{\mathrm{red}}$. With this notation we define

$$Q_\varepsilon^p \equiv \bigcup_{C \in \mathcal{C}_\varepsilon^p} Q_{\mathrm{red}}, \qquad I_\varepsilon^p \equiv \bigcup_{C \in \mathcal{C}_\varepsilon^p} I_{\mathrm{red}}.$$

Notice that $I_0 \subset\subset I_\varepsilon^0 \cup I_\varepsilon^1$.

Let us fix $\psi \in C_c^\infty((-\frac{1}{2}, \frac{1}{2}); [0,1])$ with $\psi \equiv 1$ on $(-\frac{3}{8}, \frac{3}{8})$. With ψ we define the cut-off function ψ_C^ε on $C = C_{i,j}^\varepsilon$ satisfying $\psi_C^\varepsilon = 1$ on C_{red}

$$\psi_C^\varepsilon(t,x) \equiv \psi_I^\varepsilon(t)\psi_Q^\varepsilon(x) \quad \text{with} \quad \psi_Q^\varepsilon(x) \equiv \psi_{j_1}^\varepsilon(x_1)\cdots\psi_{j_n}^\varepsilon(x_n),$$

and $\psi_{j_l}^\varepsilon(x_l) = \psi\left(\frac{x_l - \varepsilon j_l}{\varepsilon}\right)$ and $\psi_I^\varepsilon(t) = \psi\left(\frac{t-\varepsilon(i-p/2)}{\varepsilon}\right)$.

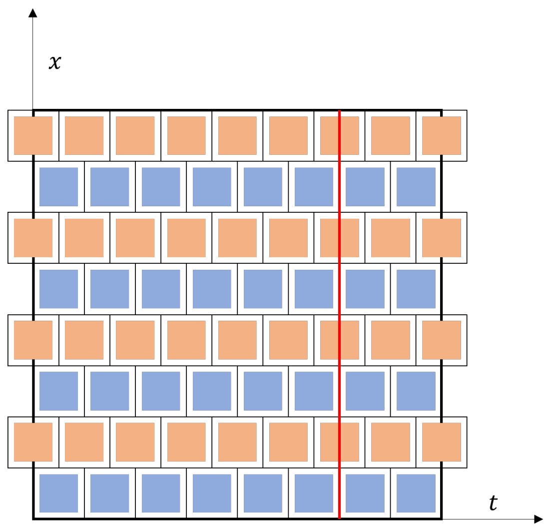


Figure 3.1: The shifted grid on $C_0 = I_0 \times Q_0$. The blue region correspond to the reduced cylinders for $p(j) = 0$ and the orange region to the reduced cylinders for $p(j) = 1$. Thus, we perturb u at each time slice (red line). These cylinders are $C_\varepsilon^{i,j}$ for $\varepsilon = 2^{-N}\ell_0$ for $N = 3$.

1.2. The sample. We define the discretization of any $f \in C(I_{\varepsilon_1} \times Q_0)$ on the grid as the simple function

$$\boxdot_\varepsilon f \equiv \sum_{C \in \mathcal{C}_\varepsilon} f(y_C)\mathbb{1}_C.$$

Thus, for any $p \in \{0,1\}$, it follows that

$$\sup_{t\in I}\left|\int_{Q_\varepsilon^p} \boxdot_\varepsilon f(t,x)\,\mathrm{d}x - \tfrac{1}{2}\left(\tfrac{3}{4}\right)^n \int_{Q_0} f(t,x)\,\mathrm{d}x\right| \to 0,$$

as $\varepsilon \to 0$. Hence, since $d \circ u$ and the constant function 1 are (uniformly) continuous on $I_{\varepsilon_1} \times Q$, there exists $N_2 \geq N_1$ depending on d, u, α and $|Q| = \ell_0^{n+1}$ satisfying

$$\sup_{t\in I}\left|\int_{Q_\varepsilon^p} \boxdot_\varepsilon d(t,x,u(t,x))\,\mathrm{d}x - \tfrac{1}{2}\left(\tfrac{3}{4}\right)^n \int_{Q_0} d(t,x,u(t,x))\,\mathrm{d}x\right| \leq \tfrac{1}{8}\left(\tfrac{3}{4}\right)^n \alpha, \tag{3.18}$$

$$\left||Q_\varepsilon^p| - \tfrac{1}{2}\left(\tfrac{3}{4}\right)^n |Q_0|\right| \leq \tfrac{1}{4}\left(\tfrac{3}{4}\right)^n |Q_0|, \tag{3.19}$$

for every $p \in \{0,1\}$ and $0 < \varepsilon \leq \varepsilon_2 = 2^{-N_2}\ell_0$.

1.3. Stable Λ-segments. We refine the grid to guarantee that we can take a single long Λ-segment for the whole $u(C)$ for each $C \in \mathcal{C}_\varepsilon$. Since u is (uniformly) continuous on $I_{\varepsilon_1} \times Q_0$, we have $\{(y, u(y)) : y \in I_{\varepsilon_1} \times Q_0\} \subset\subset U$. Hence, Lemma 3.4.3 implies that there are $\delta, \varepsilon_3(u, C_0) > 0$ so that, for all $y_0 \in I_{\varepsilon_1} \times Q_0$ there is $\underline{u}(y_0) \in \Lambda_{(y_0, u(y_0))}$ satisfying

$$\operatorname{dist}(u(y) + [-\underline{u}(y_0), \underline{u}(y_0)], \partial U_y) \geq \delta,$$

for all $y \in I_{\varepsilon_1} \times Q_0$ with $|y - y_0|_\infty \leq 2^{-N_3}\ell_0 = \varepsilon_3 \leq \varepsilon_2$.

From now on we fix ε_3 and, whenever there is no ambiguity, we will skip it to simplify the notation.

Step 2. The perturbation. For each cube $C \subset \mathcal{C}$ let us denote $u_C \equiv u(y_C)$ and $\underline{u}_C \equiv \underline{u}(y_C)$. Let $\xi_C \in \mathbb{R} \times \mathbb{S}^{n-1}$ be the direction and $u'_{k,C}$ the localized smooth solution in (H1) associated to $\underline{u}_C$ and ψ_C. Then, since

$$\begin{aligned} u(y) + u'_{k,C}(y) = u(y) + \underline{u}_C h(k\xi_C \cdot y)\psi_C(y) &\qquad \in u(y) + [-\underline{u}_C, \underline{u}_C] \\ + u'_{k,C}(y) - \underline{u}_C h(k\xi_C \cdot y)\psi_C(y) &\qquad = O(1/k) \end{aligned}$$

for all $y \in C$, there is $k_0 \in \mathbb{N}$ satisfying

$$u(y) + u'_{k,C}(y) \in U_y,$$

for all $y \in C$ and $k \geq k_0$. Summing over all $\mathcal{C}$, we define the perturbation as usual

$$u'_k \equiv \sum_{C \in \mathcal{C}} u'_{k,C} \quad \text{and} \quad u_k \equiv u + u'_k.$$

Notice that $u_k \in X_0(\bar{u})$.

We claim that $u_k \to u$ in $C_t L^p_{w^*}$. For every $v \in L^{p^*}(\mathscr{D}; \mathbb{R}^m)$ we have

$$\begin{aligned} &\left| \int_{\mathscr{D}} (u_k(t,x) - u(t,x)) \cdot v(x) \,\mathrm{d}x \right| = \sum_{C \in \mathcal{C}} \left| \int_Q u'_{k,C}(t,x) \cdot v(x) \,\mathrm{d}x \right| \\ &\leq \sum_{C \in \mathcal{C}} \Bigg(\|u'_{k,C}(t,\cdot) - \underline{u}_C h(k\xi_C \cdot (t,\cdot))\psi_C(t,\cdot)\|_{L^p} \|v\|_{L^{p^*}} \\ &\qquad + |\underline{u}_C| \left| \int_Q h(k\xi_C \cdot (t,x))\psi_Q(x) \cdot v(x) \,\mathrm{d}x \right| \Bigg) \\ &\to \sum_{C \in \mathcal{C}} |\underline{u}_C| \left| \int_Q \psi_Q(x) \cdot v(x) \,\mathrm{d}x \right| \left| \int_{\mathbb{T}} h \,\mathrm{d}\tau \right| = 0, \end{aligned}$$

uniformly in $t \in [0, T]$ as $k \to \infty$, where we have applied Lemma 3.4.2.

Step 3. The $\mathcal{F}$-property. We claim that there exists $k_1 \geq k_0$ so that $u_k \in X_0(\bar{u}, \mathcal{F})$ for all $k \geq k_1$. Since $\mathcal{F}$ is finite, we may assume w.l.o.g. that $\mathcal{F} = \{(F, g, Z, E)\}$ for simplicity.

3.1. Compare u_k with u. By adding and subtracting $F(u)$ we get

$$\begin{aligned}&\left|\fint_{Z(t,C)}(F(u_k)-F(\bar{u}))(t,x)g(t,x)\,\mathrm{d}x\right|\\ &\leq\left|\fint_{Z(t,C)}(F(u_k)-F(u))(t,x)g(t,x)\,\mathrm{d}x\right|+c_0E(t,C),\end{aligned}$$

for every convex body $(t,C)\subset\mathscr{U}'$, where $0<c_0(u)<1$ is the constant from Definition 3.3.4. Thus, it is enough to show that, for some fixed $0<c_1(u)<1-c_0$, there exists $k_1\geq k_0$ satisfying

$$\left|\fint_{Z(t,C)}(F(u_k)-F(u))(t,x)g(t,x)\,\mathrm{d}x\right|\leq c_1E(t,C), \tag{3.20}$$

for every convex body $(t,C)\subset\mathscr{U}'$ and $k\geq k_1$.

Let us use that $u_k=u$ outside $I_{\varepsilon_3}\times Q_0$. Notice that $\mathbf{Z}^{-1}(I_{\varepsilon_3}\times Q_0)\subset\subset\mathscr{U}'$, where recall $\mathbf{Z}:\mathscr{U}'\to\mathscr{U}$ is the $C_tC^1_x$-homeomorphism given by $\mathbf{Z}(t,x)=(t,Z(t,x))$. On the one hand, the distance term D of E is bounded from below by the constant

$$D_1=D_1(u,C_0,\mathcal{F})\equiv\inf\{D(t,x)\,:\,\mathbf{Z}(t,x)\in I_{\varepsilon_3}\times Q_0\}>0.$$

More precisely, every convex body $(t,C)\subset\mathscr{U}'$ with $\mathbf{Z}(t,C)\cap(I_{\varepsilon_3}\times Q_0)\neq\emptyset$ (equivalently $t\in I_{\varepsilon_3}$ and $C\cap Z^{-1}(t,Q_0)\neq\emptyset$) satisfies

$$\sup_{x\in C}D(t,x)\geq D_1,$$

while the l.h.s. of (3.20) vanishes for the remaining (t,C)'s. On the other hand, using that $\mathbf{Z}^{-1}(I_{\varepsilon_3}\times Q_0)$ is bounded, we can fix a cube $Q_0'\subset\mathbb{R}^n$ of side length ℓ_0' containing its projection into $\mathbb{R}^n$

$$\bigcup_{t\in I_{\varepsilon_3}}Z^{-1}(t,Q_0)\subset Q_0'.$$

Therefore, to prove (3.20) it is enough to check that

$$\left|\int_{Z(t,C')}(F(u_k)-F(u))(t,x)g(t,x)\,\mathrm{d}x\right|\leq c_1D_1(1\wedge|Z(t,C')|^{\gamma}), \tag{3.21}$$

for every convex body $(t,C')\subset\mathscr{U}_0'\equiv(I_{\varepsilon_3}\times Q_0')\cap\mathscr{U}'$, where we have replaced C with the convex body $C'=C\cap Q_0'$, which satisfies $|Z(t,C')|\leq|Z(t,C)|$ and $u_k=u$ on $Z(t,C\setminus C')$. Thus, from now on we will denote C instead of C' for ease of notation.

3.2. Case $Z(t,C)$ small. Let B be the compact and metrizable subset of $L^p_S(\mathscr{D})$ from Definition 3.3.1. Recall that $F\in C(L^p_S;L^{p_1}_{w^*})$ and $g\in C_tL^{p_2}$ with $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1$ and that $0\leq\gamma<\frac{1}{p_3}$. Then, for the constant

$$F_1=F_1(u,C_0,\mathcal{F})\equiv\sup_{u\in B}\|F(u)\|_{L^{p_1}}\sup_{t\in I_{\varepsilon_3}}\|g(t)\|_{L^{p_2}}<\infty,$$

any convex body $(t, C) \subset \mathscr{U}_0'$ satisfying both $|Z(t,C)| \le 1$ and $2F_1|Z(t,C)|^{\frac{1}{p_3}-\gamma} \le c_1 D_1$, Hölder's inequality implies (3.21) because

$$\int_{Z(t,C)} |(F(u_k) - F(u))(t,x)||g(t,x)| \, \mathrm{d}x \le 2F_1|Z(t,C)|^{1/p_3} \le c_1 D_1 (1 \wedge |Z(t,C)|^\gamma).$$

3.3. Case $Z(t,C)$ big. For the remaining convex bodies $(t,C) \subset \mathscr{U}_0'$ there is a constant $Z_1 = Z_1(u, C_0, \mathcal{F}) > 0$ satisfying

$$1 \wedge |Z(t,C)|^\gamma \ge Z_1. \tag{3.22}$$

Therefore, to prove (3.21) it is enough to check that

$$\left| \int_{Z(t,C)} (F(u_k) - F(u))(t,x) g(t,x) \, \mathrm{d}x \right| \le c_1 D_1 Z_1,$$

for every convex body $(t,C) \subset \mathscr{U}_0'$ satisfying (3.22). Notice that we can take a frequency k_1 for each (t,C) separately. In order to obtain an uniform k_1 we will fix finite families of representative sets and times.

3.3.1. Finite family of sets. Let us fix $N \in \mathbb{N}$ satisfying

$$2F_1 \sup_{t \in I_{\varepsilon_3}} \|\nabla Z(t)\|^{1/p_3}_{C(Z^{-1}(t,Q_0))} (4n^{3/2} 2^{-N} |Q_0'|)^{1/p_3} \le \tfrac{1}{3} c_1 D_1 Z_1. \tag{3.23}$$

With this N we construct the homogeneous grid in Q_0' with side length $2^{-N}\ell_0'$. Let $\{Q_1, \dots, Q_{j_0}\}$ be the finite family formed by all the cubes of this grid ($j_0 = 2^{Nn}$). Then, we consider the finite family formed by all the possible unions of these cubes ($Q_\emptyset = \emptyset$)

$$\{Q_J \,:\, J \subset \{1, \dots, j_0\}\} \quad \text{where} \quad Q_J \equiv \bigcup_{j \in J} Q_j.$$

Thus, for each convex body $(t,C) \subset \mathscr{U}_0'$ there is a maximal Q_J contained in C:

$$Q_J(C) \equiv \bigcup_{Q_j \subset C} Q_j.$$

Let us measure $C \setminus Q_J$. Denote $C_r \equiv \{x \in C \,:\, \mathrm{dist}(x, \partial C) \ge r\}$ by the convex subset of C for any $r \ge 0$. We claim that $C_r \subset Q_J(C)$ for $r = 2\sqrt{n}2^{-N}\ell_0' = 2\mathrm{diam}(Q_j)$ (for any j). Given $x_0 \in C_r$, let Q_j containing x_0. Then, for all $x \in Q_j$ we have

$$\mathrm{dist}(x, \partial C) \ge \mathrm{dist}(x_0, \partial C) - |x - x_0| \ge r - \tfrac{r}{2} > 0,$$

that is, $Q_j \subset C$ and so $x_0 \in Q_j \subset Q_J(C)$. Hence, by applying the monotonicity of the perimeter of convex bodies and the Fubini's theorem, it follows that

$$|C \setminus Q_J| \le |C \setminus C_r| \le |\partial C| r \le |\partial Q_0'| r = 4n^{3/2} 2^{-N} |Q_0'|, \tag{3.24}$$

where $|\partial C|$ and $|\partial Q_0'| = 2n(\ell_0')^{n-1}$ denote the perimeter of C and Q_0' respectively. In particular, by applying (3.23), (3.24) and Hölder's inequality, we get

$$\begin{aligned} \int_{Z(t, C \setminus Q_J)} |(F(u_k) - F(u))(t,x)||g(t,x)| \, \mathrm{d}x &\le 2F_1 \sup_{t \in I_{\varepsilon_3}} \|\nabla Z(t)\|^{1/p_3}_{C(Z^{-1}(t,Q_0))} |C \setminus Q_J|^{1/p_3} \\ &\le \tfrac{1}{3} c_1 D_1 Z_1, \end{aligned} \tag{3.25}$$

for every convex body $(t,C)\subset\mathscr{U}_0'$.

3.3.2. Finite family of times. Since

$$t\mapsto G_J(t,\cdot)\equiv g(t,\cdot)\mathbb{1}_{Z(t,Q_J\cap Z^{-1}(t,Q_0))}(\cdot),$$

is (uniformly) continuous from I_{ε_3} to $L^{p_1^*}$, there exists a finite family of times $\{t_1,\dots,t_{i_0}\}\subset I_{\varepsilon_3}$ such that, for each $t\in I_{\varepsilon_3}$ there is t_i satisfying

$$2\sup_{u\in B}\|F(u)\|_{L^{p_1}}\|G_J(t,\cdot)-G_J(t_i,\cdot)\|_{L^{p_1^*}}\leq\tfrac{1}{3}c_1D_1Z_1,\tag{3.26}$$

for all $J\subset\{1,\dots,j_0\}$.

3.3.3. Conclusion. Once we have fixed these finite families, since $u_k\to u$ in $C_tL^p_S$, there is $k_1\geq k_0$ satisfying

$$\sup_{t\in I_{\varepsilon_3}}\left|\int_{Q_0}(F(u_k)-F(u))(t,x)G_J(t_i,x)\,\mathrm{d}x\right|\leq\tfrac{1}{3}c_1D_1Z_1,\tag{3.27}$$

for all $i\in\{1,\dots i_0\}$, $J\subset\{1,\dots j_0\}$ and $k\geq k_1$.

Finally, for every convex body $(t,C)\subset\mathscr{U}_0'$ satisfying (3.22), take Q_J and t_i as in the *steps 3.3.1-2.* Then, by adding and subtracting first the term

$$\begin{aligned}\int_{Z(t,Q_J)}(F(u_k)-F(u))(t,x)g(t,x)\,\mathrm{d}x&=\int_{Z(t,Q_J\cap Z^{-1}(t,Q_0))}(F(u_k)-F(u))(t,x)g(t,x)\,\mathrm{d}x\\&=\int_{Q_0}(F(u_k)-F(u))(t,x)G_J(t,x)\,\mathrm{d}x,\end{aligned}$$

where we have used that $u_k=u$ outside $I_{\varepsilon_3}\times Q_0$, and secondly the term

$$\int_{Q_0}(F(u_k)-F(u))(t,x)G_J(t_i,x)\,\mathrm{d}x,$$

the inequalities (3.25)-(3.27) yield

$$\begin{aligned}&\left|\int_{Z(t,C)}(F(u_k)-F(u))(t,x)g(t,x)\,\mathrm{d}x\right|\\&\leq\left|\int_{Z(t,C\setminus Q_J)}(F(u_k)-F(u))(t,x)g(t,x)\,\mathrm{d}x\right| &&\leq\tfrac{1}{3}c_1D_1Z_1\\&+\left|\int_{Q_0}(F(u_k)-F(u))(t,x)(G_J(t,x)-G_J(t_i,x))\,\mathrm{d}x\right| &&\leq\tfrac{1}{3}c_1D_1Z_1\\&+\left|\int_{Q_0}(F(u_k)-F(u))(t,x)G_J(t_i,x)\,\mathrm{d}x\right| &&\leq\tfrac{1}{3}c_1D_1Z_1\end{aligned}$$

for all $k\geq k_1$, as we wanted.

Step 4. The β-property. We claim that (3.17) holds for

$$\beta=\min\left\{\frac{\alpha}{2},|Q_0|\frac{\|h\|_{L^r}^r}{8}\left(\frac{3}{4}\right)^n\phi^*\left(\frac{1}{|Q_0|}\frac{\alpha}{8}\left(\frac{3}{4}\right)^n\right)\right\},$$

where r is the degree of homogeneity of H, and ϕ^* is the convex-envelope (cf. [84, Def. 1.7]) of ϕ, which is also increasing with $0 < \phi^* \leq \phi$. Firstly, the convexity property (3.10) yields ($u_k = u + u'_k$)

$$\begin{aligned}&\int_{Q_0} d(t,x,u_k(t,x))\,\mathrm{d}x \\ &\leq \int_{Q_0} d(t,x,u(t,x))\,\mathrm{d}x - \int_{Q_0} G(u(t,x))\cdot u'_k(t,x)\,\mathrm{d}x - \int_{Q_0} H(u'_k(t,x))\,\mathrm{d}x.\end{aligned}$$

Since $G \circ u$ is (uniformly) continuous on $I_{\varepsilon_3} \times \bar{Q}_0$, we may assume (for a subsequence if necessary) that this linear term in u'_k vanishes as

$$\sup_{t\in I}\left|\int_{Q_0} G(u(t,x))\cdot u'_k(t,x)\,\mathrm{d}x\right| \leq \frac{1}{k}.$$

Let us split I into $I_<$ and $I_\geq$ with

$$I_{\bigcirc} := \left\{t \in I \,:\, \int_{Q_0} d(t,x,u(t,x))\,\mathrm{d}x \bigcirc \frac{\alpha}{2}\right\}.$$

Then, using that $H \geq 0$ and $J(z) \geq \alpha$, at any $t \in I_<$ we deduce that

$$\int_{Q_0} d(t,x,u_k(t,x))\,\mathrm{d}x - \frac{1}{k} \leq \frac{\alpha}{2} \leq J(z) - \beta.$$

Let us assume now that $t \in I_\geq$. In this case (3.18) yields

$$\min_{p\in\{0,1\}} \int_{Q^p} \boxdot d(t,x,u(t,x))\,\mathrm{d}x \geq \frac{\alpha}{8}\left(\frac{3}{4}\right)^n.$$

Let us analyse the term with H. Given $C \in \mathcal{C}$, since $\psi_C = 1$ on C_{red}, we have

$$H(u'_k(y)) = H(\underline{u}_C h(k\xi_C \cdot y)), \quad y \in C_{\mathrm{red}}.$$

Hence, Lemma 3.4.2 implies

$$\int_{Q_{\mathrm{red}}} H(u'_k(t,x))\,\mathrm{d}x \to \int_{Q_{\mathrm{red}}} \mathrm{d}x \int_{\mathbb{T}} H(\underline{u}_C h(\tau))\,\mathrm{d}\tau = |Q_{\mathrm{red}}| H(\underline{u}_C) \int_{\mathbb{T}} |h(\tau)|^r\,\mathrm{d}\tau,$$

uniformly in $t \in I$ as $k \to \infty$. By (H2) we have

$$H(\underline{u}_C) \geq \phi^*(d(y_C, u_C)),$$

with $d(y_C, u_C) = \boxdot d(y, u(y))$ for all $y = (t,x) \in C$. Therefore, for $p \in \{0,1\}$ so that $t \in I_p$, by summing over all the cubes Q_{red} and applying Jensen's inequality, we get

$$\begin{aligned}\lim_{k\to\infty} \int_{Q^p} H(u'_k(t,x))\,\mathrm{d}x &\geq \|h\|_{L^r}^r \int_{Q^p} \phi^*(\boxdot d(t,x,u(t,x)))\,\mathrm{d}x \\ &\geq |Q^p| \|h\|_{L^r}^r \phi^*\left(\frac{1}{|Q^p|}\int_{Q^p} \boxdot d(t,x,u(t,x))\,\mathrm{d}x\right) \\ &\geq |Q_0| \frac{\|h\|_{L^r}^r}{4}\left(\frac{3}{4}\right)^n \phi^*\left(\frac{1}{|Q_0|}\frac{\alpha}{8}\left(\frac{3}{4}\right)^n\right) \geq 2\beta,\end{aligned}$$

uniformly in $t \in I_p \cap I_{\geq}$. In general, we may assume (for a subsequence if necessary) that

$$\int_{Q_0} H(u_k'(t,x))\,\mathrm{d}x \geq \beta.$$

for all $t \in I_{\geq}$. In summary, we have seen that

$$J(u_k) - \frac{1}{k} \leq J(u) - \beta.$$

This concludes the proof. □

Corollary 3.4.1. $X_J \subset J^{-1}(0)$.

Proof. We prove it by contradiction. Assume that there is $u \in X_J$ with $J(u) > 0$. Let $(u_j) \subset X_0$ converging to u. Since J is continuous at u, we may assume (for a subsequence if necessary) that $J(u_j) \geq \frac{1}{2}J(u) \equiv \alpha > 0$. By applying Proposition 3.4.1, for each u_j there is $(u_{j,k}) \subset X_0$ converging to u_j and satisfying

$$\limsup_{k\to\infty} J(u_{j,k}) \leq J(u_j) - \beta.$$

Since X is a complete metric space, we can construct a diagonal sequence $u_{j,k(j)} \to u$ satisfying $J(u_{j,k(j)}) \nrightarrow J(u)$, which contradicts $u \in X_J$. □

Next, we are able to prove Theorem 3.3.3. This particularly proves Theorem 3.3.1 for $\mathcal{F} = \emptyset$.

Proof of Theorem 3.3.3. Let us take a countable family of cylinders $C_j \subset\subset \mathscr{U}$ satisfying $\cup_j C_j = \mathscr{U}$. Let J_j be the corresponding functional on C_j. Then, $\cap_j X_{J_j}$ is countable intersection of open and dense sets, and it is contained in $\cap_j J_j^{-1}(0)$.

Finally, by (H2) we conclude that any $u \in \cap_j J_j^{-1}(0)$ satisfies $u(y) \in K_y$ for all $y \in [0,T] \times \mathscr{D}$, and so u is a weak solution to (L, K). □

We conclude this chapter by proving Theorem 3.3.2. This is a generalization of the mixing property (1.24).

Proof of Theorem 3.3.2. Fix $C = I \times B \subset\subset \mathscr{U}$, $p^* \leq q < \infty$, $0 < s < 1$ and $N > 0$. We consider

$$W_N^{s,q}(B;\mathbb{S}^{m-1}) := \{\zeta \in L^q(B;\mathbb{S}^{m-1}) \,:\, |\zeta|_{W^{s,q}(B)} \leq N\}.$$

Recall the fractional version of the Rellich–Kondrachov theorem ([58])

$$W_N^{s,q}(B;\mathbb{S}^{m-1}) \subset\subset L^q(B;\mathbb{S}^{m-1}).$$

Let us denote $W \equiv W_N^{s,q}(B;\mathbb{S}^{m-1})$. We consider the subset of X

$$X_{C,W} := \left\{ u \in X \,:\, \int_B (a(t,x) - \pi(u(t,x)) \cdot \zeta(x))\,\mathrm{d}x = 0 \quad \text{for some} \quad (t,\zeta) \in I \times W \right\}.$$

Notice that

$$\pi(u) \cdot \zeta \leq |\pi(u)| \leq a,$$

for all $u \in X$ and $\zeta \in W$. Moreover, on $\mathscr{U}$

$$\pi(u) \cdot \zeta \leq |\pi(u)| < a,$$

for all $u \in X_0$ and $\zeta \in W$. Hence $X_{C,W} \cap X_0 = \emptyset$, which implies that $\operatorname{int}(X_{C,W}) = \emptyset$ and so

$$X \setminus X_{C,W} = \left\{ u \in X \ : \ \int_B (a(t,x) - \pi(u(t,x)) \cdot \zeta(x))\,\mathrm{d}x > 0 \quad \text{for all} \quad (t,\zeta) \in I \times W \right\},$$

is dense in X. We claim that $X_{C,W}$ is closed. Let $(u_k) \subset X_{C,W}$ converging to some $u \in X$. There are $(t_k, \zeta_k) \subset I \times W$ so that

$$\int_B (a(t_k,x) - \pi(u_k(t_k,x)) \cdot \zeta_k(x))\,\mathrm{d}x = 0.$$

Since $I \times W$ is compact, we may assume (for a subsequence if necessary) that $(t_k,\zeta_k) \to (t,\zeta) \in I \times W$. Notice that $u_k(t_k) \to u(t)$ in $L^p_{w^*}$ because

$$d_B(u_k(t_k), u(t)) \leq d_X(u_k, u) + d_B(u(t_k), u(t)) \to 0.$$

Hence, using that $a(t_k) \to a(t)$ and

$$\begin{aligned} \left| \int_B (\pi(u_k(t_k,x)) \cdot \zeta_k(x) - \pi(u(t,x)) \cdot \zeta(x))\,\mathrm{d}x \right| \leq & \left| \int_B \pi(u_k(t_k,x) - u(t,x)) \cdot \zeta(x)\,\mathrm{d}x \right| && \to 0 \\ & + \sup_k \|u_k\|_{C_t L^p} |B|^{1-(1/p+1/q)} \|\zeta_k - \zeta\|_{L^q} && \to 0, \end{aligned}$$

we deduce that

$$\int_B (a(t,x) - \pi(u(t,x)) \cdot \zeta(x))\,\mathrm{d}x = 0,$$

and so $u \in X_{C,W}$. Therefore, $X \setminus X_{C,W}$ is open and dense.

Finally, consider the intersection R_0 of all these $X \setminus X_{C,W}$ with $C = I \times B$ defined with rational parameters, and $W = W^{s,q}_N(\bar{B}; \mathbb{S}^{n-1})$ with $s, q \in \mathbb{Q}$ and $N \in \mathbb{N}$. By the Baire category theorem, this set R_0 is residual. Since the set of exact solutions contains a residual set R (with $|\pi(u)| = a$ on $\mathscr{U}$ for all $u \in R$), the set $R_{\text{wild}} = R \cap R_0$ is also residual.

We claim that any $u \in R_{\text{wild}}$ satisfies the statement of the theorem. We prove it by contradiction. Assume that there is $u \in R_{\text{wild}}$ satisfying $u(t) \in W^{s,q}(\Omega)$ for some open $(t,\Omega) \subset \mathscr{U}$, $0 < s < 1$ and $1 \leq q < \infty$. Let us take a ball $B \subset\subset \Omega$ and a time interval $I \ni t$, so that $C = I \times B \subset\subset \mathscr{U}$ is defined with rational parameters. Since $u \in R$ and $a \gg 0$ on C (so $1/a \in C_t C^1$ on C), we have

$$\zeta \equiv \frac{\pi(u)}{a}(t) \in W^{s,q}(B; \mathbb{S}^{m-1}).$$

Take rational parameters s_0, q_0 satisfying $s_0 q_0 \leq sq$ and $N \geq |\zeta|_{W^{s_0,q_0}(B)}$. Then, since $\zeta \in W^{s_0,q_0}_N(B; \mathbb{S}^{m-1})$ and

$$\int_B (a(t,x) - \pi(u(t,x)) \cdot \zeta(x))\,\mathrm{d}x = 0,$$

we deduce that $u \notin R_0$, which contradicts $u \in R_{\text{wild}}$. □

Chapter 4

Dissipative Euler flows for vortex sheets without fixed sign

This chapter presents the paper [103], joint work with László Székelyhidi.

4.1 Introduction and main results

The motion of a 2D ideal incompressible fluid is described by its velocity field $v(t,x)$, which satisfies the incompressible Euler (IE) equation for some scalar pressure $p(t,x)$

$$\partial_t v + \mathrm{div}(v \otimes v) = -\nabla p, \tag{4.1a}$$

$$\mathrm{div} v = 0, \tag{4.1b}$$

in $\mathbb{R}_+ \times \mathbb{R}^2$, evolving from a divergence-free initial datum $v^\circ(x)$

$$v|_{t=0} = v^\circ. \tag{4.2}$$

In this chapter we are interested in the dynamics of vortex sheets, this is (4.1) when the initial vorticity $\omega^\circ := \mathrm{rot} v^\circ$ is concentrated on a curve. Here we consider

$$\text{curves:} \qquad z^\circ \in C^{k_\circ+1,\delta}(\mathbb{T};\mathbb{R}^2) \quad \text{closed, simple and regular,} \tag{4.3a}$$

$$\text{vorticity strengths:} \qquad \varpi^\circ \in C^{k_\circ,\delta}(\mathbb{T};\mathbb{R}), \tag{4.3b}$$

for some $k_\circ \geq 0$ and $\delta > 0$ to be determined. We may assume w.l.o.g. that z° is the positively oriented ($\circlearrowleft$) arc-length ($|\partial_\alpha z^\circ| = 1$) parametrization, and so $\mathbb{T} := \mathbb{R}/\ell_\circ\mathbb{Z}$ with $\ell_\circ \equiv \mathrm{length}(z^\circ)$. Then, ω° is the Dirac delta

$$\omega^\circ = \varpi^\circ \delta_{z^\circ}. \tag{4.4}$$

Thus, v° is recovered from ω° by the Biot-Savart law (Prop. 2.0.1)

$$v^\circ(x)^* = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\varpi^\circ(\beta)}{x - z^\circ(\beta)}\,\mathrm{d}\beta, \qquad x \neq z^\circ(\beta). \tag{4.5}$$

This velocity is bounded, anti-holomorphic outside z° but with tangential discontinuities along z°. Due to this lack of regularity we must interpret (4.1) in its weak formulation.

Definition 4.1.1. Let us denote

$$L^\infty_{\mathrm{div}}(\mathbb{R}^2) := \left\{ v \in L^\infty(\mathbb{R}^2;\mathbb{R}^2) \,:\, \int_{\mathbb{R}^2} v \cdot \nabla\psi \,\mathrm{d}x = 0 \quad \forall \psi \in C^1_c(\mathbb{R}^2) \right\}.$$

A pair

$$(v,p) \in C([0,T]; L^\infty_{\mathrm{div}}(\mathbb{R}^2) \times L^\infty_{w^*}(\mathbb{R}^2))$$

is a **weak solution** to IE if

$$\int_0^t \int_{\mathbb{R}^2} (v \cdot \partial_t \Psi + v \otimes v : \nabla\Psi + p \,\mathrm{div}\Psi) \,\mathrm{d}x \,\mathrm{d}s = \int_{\mathbb{R}^2} v(t) \cdot \Psi(t) \,\mathrm{d}x - \int_{\mathbb{R}^2} v^\circ \cdot \Psi^\circ \,\mathrm{d}x$$

holds for every test function $\Psi \in C^1_c(\mathbb{R}^3;\mathbb{R}^2)$ and $0 \le t \le T$. In addition, (v,p) is a **strong solution** to IE if it is continuous and piecewise C^1 on $]0,T] \times \mathbb{R}^2$.

Brief background

The Cauchy problem (4.1) for the vortex sheet initial data (4.5) serves as a simplified model of many physical phenomena observed in the atmosphere and oceans related to turbulence, such as mixing layers, jets and wakes (see [96, sec. 9] and [132]). By neglecting the effects of surface tension and viscosity, this predicts the evolution of two incompressible and irrotational fluids (with the same constant densities, e.g. two masses of water) when they come into contact with different motions at z° ([10]). The (tangential) discontinuity in the velocity induces vorticity at the interface z° (4.4). Experimentally, this instability triggers a laminar-turbulent transition in a neighborhood of the vortex sheet ([44, sec. 14]). Mathematically, this Cauchy problem has been tackled from two different approaches.

One approach begins with the celebrated paper [56]. Through a compensated-compactness type argument Delort proved that, provided ω° belongs to the class $D^+(\mathbb{R}^2) := \mathcal{M}^+(\mathbb{R}^2) \cap H^{-1}(\mathbb{R}^2)$, then IE admits a global weak solution $(v,p) \in L^\infty_{\mathrm{loc}}(\mathbb{R}; L^2_{\mathrm{div,loc}}(\mathbb{R}^2) \times \mathscr{S}'(\mathbb{R}^2))$ whose vorticity $\omega(t) := \mathrm{rot} v(t)$ remains in D^+ for all times, where $\mathcal{M}$ (resp. $\mathcal{M}^+$) denotes the space of (non-negative) Radon measures. The question of uniqueness as well as how the singularity spreads is not explicit in this construction. Delort's result has been proved in different ways: by examining the concentration-cancellation effect [63, 121] and via vanishing viscosity [95, 121] and vortex [92, 120] methods. In all of them the fixed sign hypothesis seems crucial (cf. [63, sec. 5]). The case of mixed sign vortex sheets has its own interest, both for its practical applications in aerodynamics and for the complex structures created by the intertwining between regions of positive and negative vorticity ([86]). Despite this, Delort's result has been only extended to $L^p + D^+$ [56, 121, 133] and to the case of vortex sheets with reflection symmetry by Lopes, Nussenzveig and Xin [94].

The other approach aims to capture the structure of these solutions. Under the assumption that the vorticity (equivalently the discontinuity) remains concentrated on a moveable interface

$$\omega(t) = \varpi(t)\delta_{z(t)}, \tag{4.6}$$

it is classical (cf. [125, 96, 93, 25]) that the corresponding velocity field

$$v(t,x)^* = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\varpi(t,\beta)}{x - z(t,\beta)} \,\mathrm{d}\beta, \qquad x \ne z(t,\beta), \tag{4.7}$$

(with p determined by the Bernoulli's law) is a weak solution to IE if and only if

$$\varpi = \varpi^\circ + \partial_\alpha \tilde{\varpi}, \tag{4.8}$$

and $(z, \tilde{\varpi})$ solves the Birkhoff-Rott integrodifferential equations (BR)

$$\partial_t z = \mathscr{B}(\omega) + r\partial_\alpha z, \qquad \partial_t \tilde{\varpi} = r\varpi, \tag{4.9a}$$

$$z|_{t=0} = z^\circ, \qquad \tilde{\varpi}|_{t=0} = 0, \tag{4.9b}$$

where $r(t, \alpha)$ represents the re-parametrization freedom (cf. [25]) and $\mathscr{B} \equiv \mathscr{B}(\omega)$ is the Birkhoff-Rott operator

$$\mathscr{B}(t, \alpha)^* := \frac{1}{2\pi i} \mathrm{pv}\int_{\mathbb{T}} \frac{\varpi(t, \beta)}{z(t, \alpha) - z(t, \beta)} \, \mathrm{d}\beta.$$

The linear stability analysis of (4.9) w.r.t. the (steady) planar vortex sheet turns out to be an ill-posed Cauchy problem in terms of the Hilbert transform (cf. [96, sec. 9.3]). By desingularizing $\mathscr{B}$ it can be observed that the interface tends to roll-up into spiral vortices (cf. sec. 4.8). This phenomenon is known as the **Kelvin-Helmholtz** instability. The first result of local in time well-posedness of (4.9) in the class of analytic (z°, ϖ°) was given in [125] by Sulem, Sulem, Bardos and Frisch. Global in time results are only known for (not necessarily analytic) curves which are close enough to the planar vortex sheet, due to Duchon and Robert [59], by taking ϖ° somehow well prepared in terms of z°. Conversely, Caflish and Orellana exhibited in [20] examples of breakdown of analyticity in finite time (see also [107]). In addition, they proved ill-posedness in H^s for $s > 3/2$ (see also the work of Ebin and Lebeau [61, 90]). In [135] Wu proved ill-posedness in a very weak class of solutions to (4.9).

In this way, many of the data in (4.3) are not within these approaches, both for the regularity and freedom required and the possible change of sign in ϖ°.

The main results

Our aim is to present a third approach which is not only complementary to the previous two approaches but also remedies the shortcomings alluded to above. More precisely, our goal in the present chapter is to develop a robust existence theory for the incompressible Euler equation for the vortex sheet initial data (4.5) in a large class of non-analytic initial data (4.3), which at the same time is able to keep track of the geometric evolution of the vortex sheet. In our approach, following [22, 70], the sharp interface (4.6) is replaced by an opening and evolving strip, called **turbulence zone** Ω_{tur}, around the initial interface. Outside the turbulence zone the solution is analytic, but inside Ω_{tur} the weak solution is obtained using convex integration and is highly irregular. The technique we use for obtaining weak solutions follows [54, 53] and is based on the existence of a suitable **subsolution** (cf. Definition 4.2.1). The key object of study is then the subsolution, rather than the irregular weak solutions obtained by convex integration. As argued in [55, 52], the subsolution relates to the macroscopic information usually studied in connection with hydrodynamical instabilities, such as growth of the turbulent zone, geometric evolution of the instability surface, macroscopic energy transfer, etc. The main advantage of this approach is that we circumvent the classical ill-posedness and are in this way able to identify the main contributions for the evolution of the macroscopic interface.

In discussing weak solutions to the incompressible Euler equation it is important to specify some form of **admissibility** [53, 60, 91], in order to be able to rule out unphysical energy-creating solutions. A typical choice is monotonicity of the **global kinetic energy**

$$E(t) := \frac{1}{2}\int_{\mathbb{R}^2} |v(t,x)|^2 \,\mathrm{d}x.$$

This condition already suffices to guarantee weak-strong uniqueness [11, 91]. However, as in [56], for initial data as in (4.4) we expect the decay (cf. Prop. 2.0.1)

$$v(t,x)^* = \frac{1}{2\pi i x}\left(\int_{\mathbb{T}} \varpi^\circ \,\mathrm{d}\alpha + O(|x|^{-1})\right), \quad |x| \gg 1, \tag{4.10}$$

so that the total energy is not defined, unless ϖ° has zero mean. However, one may proceed as follows. First, we recall the dissipation measure associated with a weak solution to IE.

Definition 4.1.2 (Duchon, Robert [60])**.** Given a weak solution (v,p) to IE, the **dissipation** D is defined as the distribution given by

$$\partial_t e + \operatorname{div}((e+p)v) =: -D,$$

where $e := \frac{1}{2}|v|^2$ represents the kinetic energy density. Moreover, for any $0 \le t \le T$, we denote by $D(t)$ the distribution defined as

$$\langle D(t), \psi\rangle := \int_0^t \int_{\mathbb{R}^2} (e\partial_t\psi + (e+p)v\cdot\nabla\psi)\,\mathrm{d}x\,\mathrm{d}s - \int_{\mathbb{R}^2} e\psi\,\mathrm{d}x\Big|_{s=0}^{s=t}, \tag{4.11}$$

for every test function $\psi \in C_c^1(\mathbb{R}^3)$.

Under some mild conditions (cf. Lemma 4.7.1) in the limiting case $\psi \to \mathbb{1}$, (4.11) reads as

$$\int_{\mathbb{R}^2} (e(t) - e(0))\,\mathrm{d}x = -\langle D(t), \mathbb{1}\rangle,$$

which agrees with the expression

$$E(t) - E(0) = -\langle D(t), \mathbb{1}\rangle,$$

for finite-energy solutions. In this way the dissipation D allows us to formulate admissibility.

Definition 4.1.3. We say that a weak solution to IE whose dissipation D is compactly supported is **admissible** (or globally dissipative) if, for all $0 \le t \le T$,

$$\langle D(t), \mathbb{1}\rangle \ge 0.$$

For completeness, in Lemma 4.7.2 we will extend the classical weak-strong uniqueness statement to weak solutions as in Definition 4.1.3 with possibly infinite energy.

With these preparations, our main result is as follows.

Theorem 4.1.1. *Consider initial data* (4.3) *with* $k^\circ = 4$, $\delta > 0$ *and* ϖ° *not identically vanishing. There exist infinitely many admissible solutions to IE for the vortex sheet initial datum* (4.5)*.*

By [53, sec. B], these are “dissipative solutions” in the sense of Lions [91].

Evolution of the turbulence zone

By multiplying formally the momentum balance equation (4.1a) by v, it is straightforward to check that $D = 0$ wherever v is smooth enough. Indeed, the critical smoothness in 3D is the subject of the recently resolved Onsager's conjecture [114]. We refer to [13, 34, 64, 81] as well as the surveys [55, 52]. In particular, in [13, 53, 51] infinitely many admissible weak solutions with non-vanishing D were constructed. Moreover, in [48, 47, 128] it was shown that the set of initial data admitting infinitely many admissible weak solutions (called 'wild initial data') is L^2-dense.

Concrete examples of wild initial data were first found in [127], where Székelyhidi constructed weak solution to IE with decreasing E for the planar interface $z^\circ(\alpha) = (\alpha, 0)$ with vortex sheet strength $\varpi^\circ = 2$, and observed that the maximal dissipation rate $\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{1}{6}$ determines uniquely the rate of expansion $c = \frac{1}{2}$ of the turbulence zone (cf. Example 4.1.1 below). This example then served as the basis for a large number of explicit non-uniqueness examples, see e.g. [22, 26, 29, 70, 76, 99, 102, 126]. A common property of these explicit wild initial data is the fact that they are associated with an unstable interface of discontinuity.

Our construction for the incompressible Euler equation follows the approach in [22, 70]. Let us recall the geometric setup from section 2.1. At each time slice $0 < t \leq T \ll 1$, the turbulence zone $\Omega_{\mathrm{tur}}(t)$ is defined as the (open) annular region in $\mathbb{R}^2$ given by

$$\Omega_{\mathrm{tur}}(t) := \{Z(t, \alpha, \lambda) \, : \, c(\alpha) > 0, \, \lambda \in (-1, 1)\}, \tag{4.12}$$

parametrized by the map

$$Z(t, \alpha, \lambda) := z(t, \alpha) + \lambda t c(\alpha) \tau(\alpha)^\perp, \tag{4.13}$$

where z is an evolution of z°, c represents the (local) rate of expansion of the turbulence zone and τ is a unitary vector field. In order to optimize the opening of the turbulence zone, it seems suitable to consider (recall $|\partial_\alpha z^\circ| = 1$)

$$\tau := \partial_\alpha z^\circ. \tag{4.14}$$

For any fixed $\lambda \in [-1, 1]$, we will denote

$$z_\lambda := Z(\cdot, \lambda).$$

We define also the open sets $\Omega_\pm$

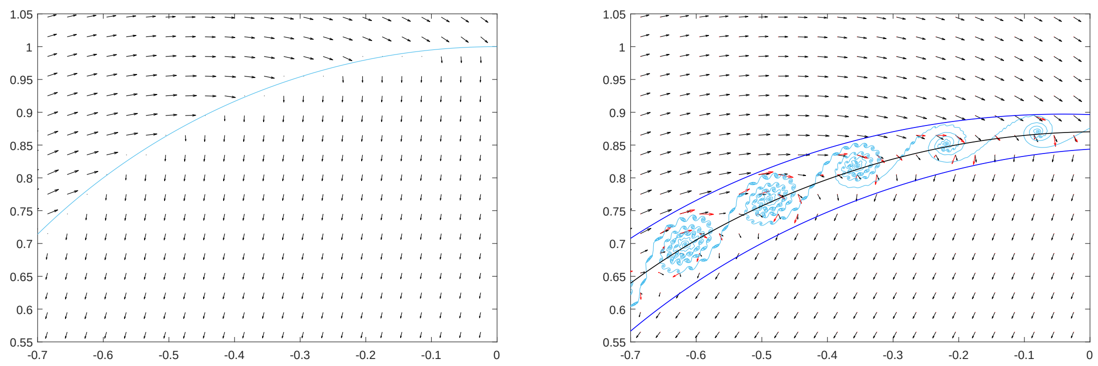
$$\begin{aligned} \Omega_-(t) &\equiv \text{ exterior domain to } z_-(t), \\ \Omega_+(t) &\equiv \text{ interior domain to } z_+(t). \end{aligned}$$

We will repeatedly use the notation $\Omega_a(t)$ with $a = \pm$, to denote the sets $\Omega_\pm(t)$. Thus, given f continuous on $\bar{\Omega}_a$ or $\bar{\Omega}_{\mathrm{tur}}$, we denote its traces on z_a as

$$f_a^\pm(t, \alpha) := \lim_{\varepsilon \to 0} f(t, z_a(t, \alpha) \pm \varepsilon \partial_\alpha z_a(t, \alpha)^\perp),$$

and the corresponding mean value and jump along z_a as

$$[\![f]\!]_a := \tfrac{1}{2}(f_a^+ + f_a^-), \qquad [f]_a := f_a^+ - f_a^-.$$



(a) Vortex sheet initial datum. (b) The turbulence zone.

Figure 4.1: (a) The divergence-free velocity field v° with vortex sheet strength ϖ° along the interface z°. (b) At some $t > 0$, the macroscopic velocity field $\bar{v}(t)$, the boundary of the turbulence zone $z_\pm(t) = z(t) \pm tc\partial_\alpha z^{\circ\perp}$ (dark blue) for some $z(t)$ (black) and $c(\alpha) \propto |\varpi^\circ(\alpha)|$. Inside $\Omega_{\mathrm{tur}}(t)$ we plot the velocity field $v(t)$ (red) w.r.t. the Kelvin-Helmholtz curve $z_{\mathrm{per}}(t)$ (light blue) which starts from a tiny perturbation of z° (cf. sec. 4.8, $\varepsilon = 0.001$).

In particular, if $[f]_a = 0$ we will write $f_a(t,\alpha) = f(t, z_a(t,\alpha))$. In this case, complex path integrals along z_a can be written as

$$\int_{z_a} f(x)\,\mathrm{d}x = \int_{\mathbb{T}} f_a \partial_\alpha z_a \,\mathrm{d}\alpha = \int f_a \partial_\alpha z_a,$$

where we abbreviate $\int = \int_{\mathbb{T}} \mathrm{d}\alpha$.

Furthermore, given $b_\pm$ continuous on $z_\pm$ we will denote

$$\langle b \rangle := \tfrac{1}{2}(b_+ + b_-), \qquad \{b\} := \tfrac{1}{2}(b_+ - b_-).$$

4.1.1 Energy dissipation rate

A new feature of our approach is that we are able to link the energy dissipation rate at the vortex sheet with the growth of the turbulence zone. This is relevant in view of the search for selection criteria among infinitely many admissible weak solutions - one that has been intensively studied in recent years is motivated by the *entropy rate admissibility criterion* introduced by Dafermos [46, 30], and amounts, in a nutshell, to identifying those weak solutions which maximize the energy dissipation rate in some sense, see [67, 30] (see also [71]). To explain this point, let us recall the construction of [127].

Example 4.1.1. Let $(\bar{v}, \bar{p}, \bar{R})$ be defined as

$$\bar{v} = (\alpha, 0), \qquad \bar{p} = \tfrac{1}{2}\alpha^2 - e, \qquad \bar{R} = \begin{pmatrix} e - \frac{1}{2}\alpha^2 & \gamma \\ \gamma & e - \frac{1}{2}\alpha^2 \end{pmatrix},$$

for some (scalar) functions α, γ, e of (t, x_2), with

$$\alpha(0, x_2) = \alpha^\circ(x_2) = \operatorname{sgn} x_2. \tag{4.15}$$

Observe that the initial datum $v^\circ(x) = \bar{v}(0,x) = (\alpha^\circ(x_2), 0)$ is a shear flow whose vorticity is concentrated on the x_1-axis z° with density $\varpi^\circ = 2$. It is straightforward to check that $\mathrm{div}\bar{v} = 0$ and $\partial_t \bar{v} + \mathrm{div}(\bar{v} \otimes \bar{v} + \bar{R}) + \nabla \bar{p} = 0$ holds with $\bar{R}$ positive semidefinite if and only if

$$\partial_t \alpha + \partial_{x_2} \gamma = 0, \quad e \geq \tfrac{1}{2}\alpha^2 + |\gamma|. \tag{4.16}$$

As noted in [127] (based on [53]; see also Theorem 6.2.1 below), under this condition $(\bar{v}, \bar{p}, \bar{R})$ is a subsolution. Moreover, if there exists a space-time open set Ω_{tur} such that $e > \frac{1}{2}\alpha^2 + |\gamma|$ inside Ω_{tur} and $e = \frac{1}{2}\alpha^2$ outside it, then for any $T > 0$ there exist infinitely many weak solutions (v, p) to IE on $[0, T]$ with $v(0, x) = v^\circ(x)$ such that, in $C_t L^1_{\mathrm{loc}}$,

$$\frac{1}{2}|v|^2 = e, \quad p = \bar{p}.$$

Let us construct some explicit examples. First of all, fix $c > 0$, let $\Omega_{\mathrm{tur}} = \{|x_2| < ct\}$ (so that $z_\lambda(t, \alpha) = (\alpha, \lambda t c)$, c.f. (4.12)-(4.13)), and set

$$\alpha = \pm 1, \quad \gamma = 0, \quad e = \tfrac{1}{2} \quad \text{on} \quad \Omega_\pm.$$

It is clear that many choices exist inside Ω_{tur} which satisfy (4.16). One simple choice is given by

$$\alpha = 0, \quad \gamma = -c, \quad e = c + e' \quad \text{in } \Omega_{\mathrm{tur}},$$

for any $e' > 0$. Observe that (α, γ) is piecewise constant and the choice $\gamma = -c$ in Ω_{tur} ensures that the jump condition along z arising from (4.16) holds. With these choices a quick calculation shows that in the limiting case $e' \to 0$ the dissipation (c.f. Definition 4.1.2) of the weak solutions (v, p) so obtained satisfies

$$\langle D(t), \psi \rangle = c(\tfrac{1}{2} - c) \sum_{\lambda = \pm 1} \int_0^t \int \psi(s, z_\lambda(s, \alpha)) \, \mathrm{d}\alpha \, \mathrm{d}s, \tag{4.17}$$

so that in particular

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle D(t), \psi \rangle \big|_{t=0} = 2c(\tfrac{1}{2} - c) \int \psi(0, \alpha, 0) \, \mathrm{d}\alpha.$$

Observe that in this example the vorticity of the corresponding $\bar{v}(t)$ is concentrated on the two lines $\{x_2 = \pm ct\}$. Analogusly, if we define α, γ inside Ω_{tur} in such a way that the vorticity of $\bar{v}$ is concentrated on the $2N$ lines $\{x_2 = \lambda_j ct\}$ with $\lambda_{\pm j} = \pm \frac{2|j|-1}{2N-1}$ for $j = 1, \ldots, N$, we obtain the piecewise constant solution

$$\alpha(t, x_2) = \begin{cases} \pm \frac{j}{N}, & \lambda_j ct < \pm x_2 < \lambda_{j+1} ct, \quad j = 1, \ldots, N-1, \\ 0, & |x_2| < \lambda_1 ct, \end{cases}$$

with piecewise constant γ determined uniquely by the jump conditions arising from (4.16), namely $\gamma = -\frac{N}{2N-1} c(1 - \alpha^2)$, and $e = \frac{1}{2}\alpha^2 + |\gamma| + e'$ for some $e' > 0$. We note in passing that such construction was used in the context of the compressible Euler system under the name "fan subsolution", see [29]. In this case it can be checked that, with $[\cdot]_\lambda$ denoting the jump along $\{x_2 = \lambda ct\}$, as $e' \to 0$ we have

$$\langle D(t), \psi \rangle = \tfrac{N}{2N-1} c \left(\tfrac{2N-1}{2N} - c \right) \sum_{1 \leq |j| \leq N} \int_0^t \int \lambda_j [\alpha^2]_{\lambda_j} \psi(s, z_{\lambda_j}(s, \alpha)) \, \mathrm{d}\alpha \, \mathrm{d}s.$$

Indeed, by the definition of λ_j and α,

$$\lambda_j[\alpha^2]_{\lambda_j} = \tfrac{(2j-1)^2}{(2N-1)N^2} > 0, \quad 1 \leq |j| \leq N.$$

Hence, by applying Faulhaber's formula we get

$$\sum_{1\leq|j|\leq N} \lambda_j[\alpha^2]_{\lambda_j} = \tfrac{2}{3}\tfrac{2N+1}{N},$$

which allows to compute the associated energy dissipation rate at the initial time

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle D,\psi\rangle\big|_{t=0} = 2\bar{a}_N c(2\bar{c}_N - c)\int \psi(0,\alpha,0)\,\mathrm{d}\alpha, \tag{4.18}$$

where

$$\bar{a}_N := \frac{1}{3}\frac{2N+1}{2N-1}, \qquad \bar{c}_N := \frac{2N-1}{4N}. \tag{4.19}$$

Finally, the limiting case $N \to \infty$ can be understood as distributing the vorticity on the whole turbulence zone. This corresponds to the rarefaction wave solution (c.f. [127])

$$\alpha(t,x_2) = \frac{x_2}{ct}, \quad |x_2| < ct,$$

with $\gamma = -\frac{1}{2}c(1-\alpha^2)$. Observe that α, γ are continuous now. In this case, as $e' \to 0$ we have

$$\partial_t e = \tfrac{1}{2}(1-c)\partial_t(\alpha^2),$$

and hence the corresponding dissipation satisfies

$$\langle D(t),\psi\rangle = \tfrac{1}{2}(1-c)\int_0^t\int\int_{\{|x_2|<sc\}}[\partial_t(\alpha^2) - \tfrac{1}{2}v\cdot\nabla(\alpha^2)]\psi\,\mathrm{d}x_2\,\mathrm{d}x_1\,\mathrm{d}s,$$

which allows to compute the associated energy dissipation rate at the initial time

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle D,\psi\rangle\big|_{t=0} = \tfrac{2}{3}c(1-c)\int \psi(0,\alpha,0)\,\mathrm{d}\alpha,$$

which is indeed the limit of (4.18) as $N \to \infty$. In particular, the maximum $\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{1}{6}$ is achieved at $c = \frac{1}{2}$.

In this chapter we are considering that both ϖ° and c could depend on α. Inspired by (4.18) and recalling that in that example $\varpi^\circ(\alpha) = 2$, we introduce the following functional.

Definition 4.1.4. For given $\varpi^\circ, c \in C(\mathbb{T};\mathbb{R})$, any $N \in \mathbb{N}$ and any interval $I \subset \mathbb{T}$ we define the energy dissipation functional

$$W_I^{(N)}(c) := \bar{a}_N \int_I c(\alpha)|\varpi^\circ(\alpha)|(\bar{c}_N|\varpi^\circ(\alpha)| - c(\alpha))\,\mathrm{d}\alpha, \tag{4.20}$$

with $\bar{a}_N$, $\bar{c}_N$ given in (4.19).

It turns out that global admissibility, as in Definition 4.1.3 leads to the following relation.

Theorem 4.1.2. *For any $N \in \mathbb{N}$ there exist admissible weak solutions as in Theorem 4.1.1, such that the global rate of dissipation and expansion of Ω_{tur} are related via*

$$\int_{\mathbb{R}^2} \frac{e(t_2)-e(t_1)}{t_2-t_1}\,\mathrm{d}x = -W_{\mathbb{T}}^{(N)}(c) + O(t_2), \tag{4.21}$$

for every $0 \leq t_1 < t_2 \leq T$.

In particular, for zero-mean ϖ°'s, (4.21) yields

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -W_{\mathbb{T}}^{(N)}(c) + O(t).$$

Note that the functional $W_{\mathbb{T}}^{(N)}$ is strictly concave and has a unique global maximum (cf. Lemma 4.3.1) at $c_{\max}^{(N)} := \frac{1}{2}\bar{c}_N|\varpi^\circ|$, with

$$\max W_{\mathbb{T}}^{(N)} = W_{\mathbb{T}}^{(N)}(c_{\max}^{(N)}) = \frac{1}{48}\left(1 - \frac{1}{(2N)^2}\right)\int_{\mathbb{T}} |\varpi^\circ|^3\,\mathrm{d}\alpha.$$

Thus, by considering $N \to \infty$ we reach the maximal initial dissipation rate $\frac{\mathrm{d}E}{\mathrm{d}t}\big|_{t=0} = -\frac{1}{48}\|\varpi^\circ\|_{L^3}^3$ obtainable by our method, which agrees with [127].

The relationship (4.21) shows that the growth rate of the turbulence zone c cannot be arbitrarily large, but does not give precise information about its local growth. For this reason, we test D with a larger class of ψ's, instead of just $\psi = \mathbb{1}$. As we shall see in section 4.3 (cf. Propositions 4.3.1-4.3.3), the dissipation D from Theorem 4.1.2 belongs to $\mathcal{M}_c([0,T]\times\mathbb{R}^2)$ with $\operatorname{supp} D \subset \bar{\Omega}_{\mathrm{tur}}$, so we can extend the space of test functions ψ to indicator functions. With this notion we obtain a local version of Theorem 4.1.2.

Theorem 4.1.3. *Given $0 < \varepsilon \leq \ell_\circ$, let (4.3) with $k_\circ = 4$, $\delta > 0$ and $|\varpi^\circ| > 0$ a.e. Then, for any $N \in \mathbb{N}$ there exist infinitely many dissipative solutions to IE for the vortex sheet initial datum (4.5) so that the local rate of dissipation and expansion of Ω_{tur} are related via*

$$\left\langle \frac{D(t_2)-D(t_1)}{t_2-t_1}, \psi_I \right\rangle = W_I^{(N)}(c) + O(t_2), \tag{4.22}$$

where $\psi_I(t,x) := \mathbb{1}_{Z(t,I\times[-1,1])}(x)$, for every interval $I \subset \mathbb{T}$ with $|I| \geq \varepsilon$ and $0 \leq t_1 < t_2 \leq T$.

Since $\psi_{\mathbb{T}} = \mathbb{1}_{\bar{\Omega}_{\mathrm{tur}}}$, Theorem 4.1.3 generalizes Theorem 4.1.2 when $\varepsilon = \ell_\circ$. For small times, (4.22) can be viewed as testing D with (space-time) cylinders $[0,T]\times B_r(z^\circ(\alpha))$ for every $\alpha \in \mathbb{T}$ and $r \gtrsim \varepsilon$, thus preventing local creation of kinetic energy along z° on length scales $\gtrsim \varepsilon$.

One possible choice of c in Theorem 4.1.3 is

$$c = s\bar{c}_N|\varpi^\circ| * \eta_\epsilon, \tag{4.23}$$

for any $0 < s < 1$ and some $\epsilon(\varpi^\circ,\varepsilon,s) \geq 0$, where (η_ϵ) is a standard mollifier (cf. Lemma 4.3.2). In particular, the dissipation rate is maximazed at $s = \frac{1}{2}$ as $\epsilon \to 0$.

It would be interesting to explore the question whether the bounds obtained in Theorems 4.1.2-4.1.3 for the energy dissipation rate are optimal. Also, a natural and very interesting question is whether one can show convergence of the vortex blob approximation [19] or the more recent approximation as a vortex layer [18] in a suitable weak sense to an admissible subsolution as in Example 4.1.1, see also Definition 4.2.1. Indeed, the analysis performed here shows certain similarities to those in [18], especially concerning the geometric setup.

These results admit some improvements and generalizations.

- The weak solutions from Theorems 4.1.1-4.1.3 belong to the stronger class $C_tL^q_{\mathrm{loc}}$ for all $1 < q < \infty$. In particular, for zero-mean ϖ°'s, also $v \in C_tL^q$.

- The regularity required in Theorems 4.1.1-4.1.3 is used to control, with relatively simple estimates, $\|z\|_{C^{2,\delta}}$ and $\|\varpi\|_{C^{1,\delta}}$. Thus, a finer analysis of section 4.4 may reduce $k_\circ$.

At the same time there are several shortcomings.

- Our results are local in time. The time of existence T in Theorems 4.1.1-4.1.2 depends on $\|z^\circ\|_{C^{k_\circ+1,\delta}}$, $\|\varpi^\circ\|_{C^{k_\circ,\delta}}$ and the chord-arc constant $\mathcal{C}(z^\circ)$. The time $T_\varepsilon \leq T$ for which we can guarantee (4.22) satisfies (see Lemma 4.3.2)
$$T_\varepsilon \gtrsim \min_{|I|\geq\varepsilon} \fint_I |\varpi^\circ|^3 \,\mathrm{d}\alpha.$$
In addition, our T_ε depends on $1/c \sim 1/(|\varpi^\circ| * \eta_{\epsilon(\varepsilon)})$. Thus, $T_\varepsilon \gg 0$ independently of ε provided $|\varpi^\circ| \gg 0$. Otherwise, more terms should be controlled in section 4.3 to avoid $T_\varepsilon \to 0$ as $\varepsilon \to 0$.

- Although $\omega^\circ \in D^+$ when $\varpi^\circ \geq 0$, in general for $t > 0$ we only know that $\omega(t)$ is a distribution, although we can guarantee at least the vorticity balance
$$\langle \omega(t), \mathbb{1} \rangle = \int \omega^\circ.$$

Organization of the chapter. We start section 4.2 recalling the concept of subsolution as well as the subsolution criterion for IE. After that, we establish the conditions under which a subsolution exists. In addition, we determine in section 4.3 the dissipation D of the weak solutions obtained via convex integration applied to the subsolution. In section 4.4 we analyse the corresponding Birkhoff-Rott type operator. Finally, we prove in section 4.5 the Theorems 4.1.1-4.1.3. The parameter N in Theorems 4.1.2-4.1.3 relates to the following ansatz: the initial vortex sheet (4.4) is split at time $t = 0$ into a sum of $2N$ vortex sheets, which separate at linear speeds for $t > 0$. The outmost curves form the boundaries of the turbulence zone Ω_{tur}. In sections 4.2-4.5 we will concentrate on the case $N = 1$ for simplicity, and will show how to extend these considerations to general N in section 4.6. Finally, we provide some pictures of how these solutions may look like in the Appendix 4.8.

4.2 The subsolution

Our weak solutions to IE are obtained from a subsolution $(\bar{v}, \bar{p}, \bar{R})$ via convex integration and indeed, the key point in our approach is to construct the subsolution. This is in alignment with the classical approach to turbulent flows via the Reynolds decomposition, splitting the velocity field into $v = \bar{v} + v'$ (and $p = \bar{p} + p'$) where $\bar{v}$ represents a mean velocity and v' the corresponding fluctuation (cf. [55]). Then, formally $(\bar{v}, \bar{p}, \bar{R})$ solves

$$\partial_t \bar{v} + \mathrm{div}(\bar{v} \otimes \bar{v} + \bar{R}) + \nabla \bar{p} = 0, \tag{4.24a}$$
$$\mathrm{div}\bar{v} = 0, \tag{4.24b}$$

where

$$\bar{R} = \overline{v \otimes v} - \bar{v} \otimes \bar{v} = \overline{v' \otimes v'}$$

is the **Reynolds stress tensor**, which satisfies $\bar{R} \geq 0$ (positive semidefinite). Observe in particular that $\frac{1}{2}\mathrm{tr}\bar{R} = e - \frac{1}{2}|\bar{v}|^2$, with $e = \frac{1}{2}|v|^2$ being the kinetic energy density (cf. Example 4.1.1). Observe that in (4.24a) the term $\mathrm{tr}\bar{R}$ may be absorbed in the pressure $\bar{p}$. A subsolution is then defined as follows.

Definition 4.2.1. A triple

$$(\bar{v}, \bar{p}, R) \in C([0,T]; L^\infty_{\mathrm{div}}(\mathbb{R}^2) \times L^\infty_{w^*}(\mathbb{R}^2) \times L^\infty_{w^*}(\mathbb{R}^2; \mathbb{R}^{2\times 2}_{\mathrm{sym}}))$$

is a weak solution to the incompressible Euler-Reynolds (IER) equation if

$$\int_0^t \int_{\mathbb{R}^2} (\bar{v} \cdot \partial_t \Psi + \sigma : \nabla \Psi) \,\mathrm{d}x\,\mathrm{d}s = \int_{\mathbb{R}^2} \bar{v}(t) \cdot \Psi(t) \,\mathrm{d}x - \int_{\mathbb{R}^2} v^\circ \cdot \Psi^\circ \,\mathrm{d}x \tag{4.25}$$

holds for every test function $\Psi \in C^1_c(\mathbb{R}^3; \mathbb{R}^2)$ and $0 \leq t \leq T$, being $\sigma \equiv \bar{v} \otimes \bar{v} + R + \bar{p}I$.
Let Ω_{tur} be an open subset of $[0,T] \times \mathbb{R}^2$ and $e \in C^0(\Omega_{\mathrm{tur}}; \mathbb{R}_+)$. A triple $(\bar{v}, \bar{p}, R)$ is a strict **subsolution** to IE w.r.t. e if $(\bar{v}, \bar{p}, R)$ is a weak solution to IER satisfying:

(i) $R = 0$ outside Ω_{tur},

(ii) $(\bar{v}, R)$ are continuous in Ω_{tur} and

$$\tfrac{1}{2}|\bar{v}|^2 + |\mathring{R}| < e.$$

Observe that (ii) is equivalent to $\bar{R} := \mathring{R} + (e - \frac{1}{2}|\bar{v}|^2)\mathrm{Id} > 0$ in Ω_{tur}. For simplicity of notation we extend e as $\frac{1}{2}|\bar{v}|^2$ outside Ω_{tur}.

We recall the following result from [53], guaranteeing the existence of (infinitely many) weak solutions based on a strict subsolution.

Theorem 4.2.1 (Subsolution criterion [53])**.** *Suppose there exists a subsolution $(\bar{v}, \bar{p}, R)$ to IE w.r.t. some $e \in C^0(\Omega_{\mathrm{tur}}; \mathbb{R}_+)$ with Ω_{tur} given by (4.12). Then, there exist infinitely many weak solutions (v, p) to IE with v satisfying*

$$\begin{aligned} v &= \bar{v} \quad \textit{outside } \Omega_{\mathrm{tur}}, \\ \tfrac{1}{2}|v|^2 &= e \quad \textit{in } \Omega_{\mathrm{tur}}, \end{aligned}$$

and p given by

$$p = \bar{p} + \tfrac{1}{2}(|\bar{v}|^2 + \mathrm{tr}R) - e. \tag{4.26}$$

Moreover, by the macroscopic h-principle (Theorem 3.3.3) we may in addition ensure that, for any fixed $\varepsilon > 0$, $g \in L^2(\mathbb{R}^2)$ and $\mathcal{T} \in C(]0,T];]0,1])$,

$$\left| \fint_{Z(t, I \times [-1,1])} (v - \bar{v})(t,x) g(x) \,\mathrm{d}x \right| \leq \mathcal{T}(t), \tag{4.27}$$

for every interval $I \subset \mathbb{T}$ with $|I| \geq \varepsilon$ and $0 \leq t \leq T$.

By this result, the construction of weak solutions as in Theorem 4.1.1 is reduced to constructing suitable subsolutions $(\bar{v}, \bar{p}, R)$ adapted to Ω_{tur}. In the following sections 4.2-4.2 we introduce our *ansatz* for the velocity $\bar{v}$, define the corresponding pressure $\bar{p}$ and derive conditions (see Proposition 4.2.4) under which a Reynolds stress leading to a strict subsolution exists.

The velocity

Following [70], our central *ansatz* is that the vorticity of the subsolution $\bar{\omega}(t) := \operatorname{rot}\bar{v}(t)$ is concentrated on the boundary of the turbulence zone

$$\bar{\omega}(t) := \frac{1}{2}\sum_{b=\pm}\varpi_b(t)\delta_{z_b(t)}, \tag{4.28}$$

with the vortex sheet strengths ϖ_b to be determined. As in (4.8), it is convenient (in fact necessary) to write it as

$$\varpi_b = \varpi^\circ + \partial_\alpha\tilde{\varpi}, \tag{4.29}$$

with $\tilde{\varpi}$ to be determined. We note in passing that one could also consider different $\tilde{\varpi}_b$ for $b = \pm$, but for our purposes this additional freedom of choice is not needed.

Thus, for $t > 0$ the velocity is recovered through the Biot-Savart law (Prop. 2.0.1)

$$\bar{v}(t,x)^* = \frac{1}{2}\sum_{b=\pm}\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{\varpi_b(t,\beta)}{x - z_b(t,\beta)}\,\mathrm{d}\beta, \qquad x \neq z_b(t,\beta), \tag{4.30}$$

which is the unique distributional solution to $\operatorname{div}\bar{v} = 0$ and $\operatorname{rot}\bar{v} = \bar{\omega}$ for (4.28) vanishing at infinity. As we shall see in section 4.4, $\bar{v}(t)$ is bounded, anti-holomorphic outside $z(t)$ but with tangential discontinuities along $z(t)$. Indeed, by the classical Sokhotski-Plemelj theorem (Prop. 2.0.2) these limits $\bar{v}_a^\pm(t,\alpha)$ are

$$\bar{v}_a^\pm = \mathscr{B}_a \mp \frac{1}{4}\frac{\varpi_a}{\partial_\alpha z_a^*}, \tag{4.31}$$

where $\mathscr{B}_a \equiv \mathscr{B}_a(\omega)$ are the Birkhoff-Rott type operators

$$\mathscr{B}_a(t,\alpha)^* = \frac{1}{2}\sum_{b=\pm}\frac{1}{2\pi i}\mathrm{pv}\int_{\mathbb{T}}\frac{\varpi_b(t,\beta)}{z_a(t,\alpha) - z_b(t,\beta)}\,\mathrm{d}\beta. \tag{4.32}$$

Notice that the pv is not necessary for $a \neq b$ when $t > 0$. Therefore, the mean value and jump of $\bar{v}$ along z_a are

$$[\![\bar{v}]\!]_a = \mathscr{B}_a, \qquad [\bar{v}]_a = -\frac{1}{2}\frac{\varpi_a}{\partial_\alpha z_a^*}. \tag{4.33}$$

Helmholtz decomposition of $\bar{v}$

It is well known that an incompressible and irrotational vector field on a simply connected domain can be expressed as the gradient of an harmonic function. However, since $\Omega_-(t)$ and $\Omega_{\mathrm{tur}}(t)$ are not simply connected, we must add the corresponding circulation. This expression is indeed necessary to recover the pressure $\bar{p}$ outside z.

Definition 4.2.2. Let $\Omega \subset \mathbb{C}$ be open and $f \in C(\Omega;\mathbb{C})$. Given a closed, positively oriented, simple curve $\gamma \in C^1(\mathbb{T};\mathbb{C})$, the **circulation** of f around γ is defined as

$$\mathscr{C}_\gamma(f) := \int_\gamma f\cdot\mathrm{d}x.$$

In particular, the **index** of $x_0 \in \mathbb{C} \setminus \gamma(\mathbb{T})$ w.r.t. γ is defined as ($K_{x_0} \equiv K(\cdot - x_0)$)

$$\mathrm{Ind}_\gamma(x_0) := \mathscr{C}_\gamma(K_{x_0}^*) = \left(\int_\gamma K_{x_0}\,\mathrm{d}x\right)_1 = \frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}x}{x - x_0} = \begin{cases} 1, & x_0 \text{ “inside” } \gamma(\mathbb{T}), \\ 0, & x_0 \text{ “outside” } \gamma(\mathbb{T}). \end{cases}$$

Proposition 4.2.1. *Let $\omega \in \mathcal{M}_c(\mathbb{R}^2)$. Then, for every γ as in Definition 4.2.2 with $\gamma(\mathbb{T}) \subset \mathbb{C} \setminus \mathrm{supp}\,\omega$, we have*

$$\mathscr{C}_\gamma((K * \omega)^*) = \omega(\Gamma_\gamma),$$

where $\Gamma_\gamma \equiv \{x \in \mathrm{supp}\,\omega \,:\, \mathrm{Ind}_\gamma(x) = 1\}$.

Proof. Denote $v = (K * \omega)^*$. On the one hand

$$\int_\gamma v\,\mathrm{d}x^* = \int_\gamma v \cdot \mathrm{d}x + i\int_\gamma v \cdot \mathrm{d}x^\perp = \mathscr{C}_\gamma(v),$$

where the last equality follows from Gauss divergence theorem because $\mathrm{div} v = 0$. On the other hand, Fubini's theorem and Cauchy's integral formula yield

$$\int_\gamma v\,\mathrm{d}x^* = \left(\frac{1}{2\pi i}\int_\gamma\int_{\mathrm{supp}\,\omega} \frac{\mathrm{d}\omega(y)}{x - y}\,\mathrm{d}x\right)^* = \left(\frac{1}{2\pi i}\int_{\mathrm{supp}\,\omega}\int_\gamma \frac{\mathrm{d}x}{x - y}\,\mathrm{d}\omega(y)\right)^* = \int_{\Gamma_\gamma}\mathrm{d}\omega(y),$$

as we wanted to prove. ☐

Given $x \in \mathbb{R}^2$ let us denote L_x by the half-line $x + (-\infty, 0]$. Let us fix $x_0 \in \Omega_+(0)$ so that $L_{x_0} \cap z^\circ(\mathbb{T})$ consists of a single point and consequently for $0 \leq t \leq T_0$ small enough $L_{x_0} \cap z_a(t, \mathbb{T}) = \{z_a(t, \alpha_{t,a})\}$ and $\Omega_r(t) \setminus L_{x_0}$ is simply connected for each region $r = +, -, \mathrm{tur}$. Fix also some $x_{t,r} \in \Omega_r(t) \setminus L_{x_0}$.

By Proposition 4.2.1, the circulation $\mathscr{C}_\gamma$ of $\bar{v} = (K * \bar{\omega})^*$ around any $\gamma \subset \Omega_r(t)$ $\circlearrowleft$-surrounding x_0 is constant on $\Omega_r(t)$, for each region $r = +, -, \mathrm{tur}$. Moreover, since $\int \varpi(t) = \int \varpi^\circ$ by (4.29), it is in fact constant on Ω_r with

$$\mathscr{C}_\gamma(\bar{v}) = \mathscr{C}_r := \begin{cases} 0 & r = +, \\ \frac{1}{2}\int \varpi^\circ & r = \mathrm{tur}, \\ \int \varpi^\circ & r = -. \end{cases} \tag{4.34}$$

Thus, we deduce that there exists a (piecewise) harmonic function $\phi(t)$ so that

$$\bar{v} = \nabla\phi + \mathscr{C} K_{x_0}^*, \tag{4.35}$$

where $\mathscr{C} = \mathscr{C}(t, x)$ is defined to be the step function taking the value $\mathscr{C}_r$ whenever $x \in \Omega_r(t)$ for $r = +, -, \mathrm{tur}$. Indeed, by this choice we obtain, for any $\gamma \in C^1(\mathbb{T}; \Omega_r(t))$ surrounding x_0

$$\mathscr{C}_\gamma(\bar{v} - \mathscr{C} K_{x_0}^*) = 0,$$

since $\mathscr{C}_\gamma(K_{x_0}^*) = \mathrm{Ind}_\gamma(x_0) = 1$. Hence ϕ can be recovered via

$$\begin{aligned} \phi(t, x) - \phi_{x_{t,r}} &:= \int_\gamma (\bar{v} - \mathscr{C} K_{x_0}^*)(t, y) \cdot \mathrm{d}y \\ &= \frac{1}{2}\left(\sum_{a=\pm} \frac{1}{2\pi i}\int_{\mathbb{T}} \varpi_a(t, \alpha)\int_\gamma \left(\frac{1}{y - z_a(t, \alpha)} - \frac{1 - \mathrm{Ind}_{z_a(t)}(x)}{y - x_0}\right)\mathrm{d}y\,\mathrm{d}\alpha\right)_1, \end{aligned} \tag{4.36}$$

for any path $\gamma \in C^1([0,1];\Omega_r(t)\setminus L_{x_0})$ with $\gamma(0)=x_{t,r}$ and $\gamma(1)=x$, where $\phi(t,x_{t,r})=\phi_{x_{t,r}}$ may be chosen. By definition (4.34)(4.36), $\phi(t,\cdot)$ is continuous on $\Omega_r(t)$ and verifies (4.35). Hence, $\phi(t,\cdot)$ is harmonic. On the one hand,

$$\int_\gamma \frac{\mathrm{d}y}{y-z_a(t,\alpha)} = \mathrm{L}_{z_a(t,\alpha)}(y)\Big|_{y=x_{t,r}}^{y=x}$$

where $\mathrm{L}_{z_a(t,\alpha)}$ is the branch of the logarithm given by the ray $l_{z_a(t,\alpha)}:[\alpha,\infty)\to\mathbb{C}$

$$l_{z_a(t,\alpha)}(\alpha') = \begin{cases} z_a(t,\alpha'), & \alpha'\in[\alpha,\alpha_{t,a}),\\ z_a(t,\alpha_{t,a})+(\alpha_{t,a}-\alpha'), & \alpha'\in[\alpha_{t,a},\infty).\end{cases}$$

On the other hand,

$$\int_\gamma \frac{\mathrm{d}y}{y-x_0} = \mathrm{Log}(y-x_0)\Big|_{y=x_{t,r}}^{y=x}$$

where Log is the principal branch of the logarithm. Hence, (4.36) reads as

$$\phi(t,x)=\frac{1}{2}\left(\sum_{a=\pm}\frac{1}{2\pi i}\int_{\mathbb{T}}\varpi_a(t,\alpha)(\mathrm{L}_{z_a(t,\alpha)}(x)-(1-\mathrm{Ind}_{z_a(t)}(x))\mathrm{Log}(x-x_0))\,\mathrm{d}\alpha\right)_1 + O(t),$$

where $O(t)$ is an arbitrary function of t. In particular, by (4.29) we deduce

$$\partial_t\phi(t,x)=\frac{1}{2}\left(\sum_{a=\pm}\frac{1}{2\pi i}\int_{\mathbb{T}}\left(\partial_{\alpha t}\tilde{\varpi}(t,\alpha)\mathrm{L}_{z_a(t,\alpha)}(x)-\varpi_a(t,\alpha)\frac{\partial_t z_a(t,\alpha)}{x-z_a(t,\alpha)}\right)\mathrm{d}\alpha\right)_1+\partial_t O(t).$$

Hence, integrating by parts, we can choose $O(t)$ in such a way that

$$\partial_t\phi(t,x)=\frac{1}{2}\left(\sum_{a=\pm}\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{(\partial_t\tilde{\varpi}\partial_\alpha z_a-\varpi\partial_t z_a)(t,\alpha)}{x-z_a(t,\alpha)}\,\mathrm{d}\alpha\right)_1. \tag{4.37}$$

Then, using the Sokhotski-Plemelj theorem, we deduce that

$$[\partial_t\phi]_a=-\frac{1}{2}\left(\frac{\partial_t\tilde{\varpi}\partial_\alpha z_a-\varpi\partial_t z_a}{\partial_\alpha z_a}\right)_1=\frac{1}{2}\left(\varpi\partial_t z_a\cdot\frac{\partial_\alpha z_a}{|\partial_\alpha z_a|^2}-\partial_t\tilde{\varpi}\right). \tag{4.38}$$

The pressure

We define $\bar{p}$ outside $z_+\cup z_-$ by means of the Bernoulli's law

$$\bar{p}:=-\partial_t\phi-\tfrac{1}{2}|\bar{v}|^2 \quad \text{outside} \quad z_+\cup z_-, \tag{4.39}$$

with ϕ given in (4.36). Since $\bar{v}$ is div-rot free outside $z_+\cup z_-$, a simple computation yields

$$\mathrm{div}(\bar{v}\otimes\bar{v})=\tfrac{1}{2}\nabla(|\bar{v}|^2) \quad \text{outside} \quad z_+\cup z_-.$$

Thus, by applying ∇ on (4.39) we deduce

$$\partial_t\bar{v}+\mathrm{div}(\bar{v}\otimes\bar{v})+\nabla\bar{p}=0 \quad \text{outside} \quad z_+\cup z_-. \tag{4.40}$$

Proposition 4.2.2. *The jump in $\bar{p}$ along z_a is*

$$[\bar{p}]_a = \frac{1}{2}\left(\partial_t\tilde{\varpi} - \varpi(\partial_t z_a - \mathscr{B}_a)\cdot\frac{\partial_\alpha z_a}{|\partial_\alpha z_a|^2}\right). \tag{4.41}$$

Proof. By applying (4.38) coupled with (4.33) and

$$\tfrac{1}{2}[|\bar{v}|^2]_a = [\![\bar{v}]\!]_a\cdot[\bar{v}]_a = -\frac{1}{2}\varpi\mathscr{B}_a\cdot\frac{\partial_\alpha z_a}{|\partial_\alpha z_a|^2}, \tag{4.42}$$

Bernoulli's law (4.39) implies (4.41). □

The Reynolds stress

Proposition 4.2.3. *Let $(\bar{v},\bar{p})$ given by (4.30) and (4.39) respectively. Then, $(\bar{v},\bar{p},R)$ is a weak solution to IER with $R=0$ outside Ω_{tur} if and only if, at each time slice $t>0$, $R=R(t,x)$ solves*

$$\mathrm{div}R = 0 \qquad in \quad \Omega_{\mathrm{tur}}, \tag{4.43a}$$

$$\pm(R\partial_\alpha z^\perp)_\pm = ib_\pm \qquad on \quad z_\pm, \tag{4.43b}$$

where, for $a=\pm$, b_a are the boundary conditions

$$b_a = \tfrac{1}{2}(\partial_t\tilde{\varpi}\partial_\alpha z_a - \varpi(\partial_t z_a - \mathscr{B}_a)). \tag{4.44}$$

Proof. Let us parametrize z_a by the map

$$\mathbf{Z}_a(t,\alpha) := (t, z_a(t,\alpha)).$$

Then, since

$$\partial_t\mathbf{Z}_a\times\partial_\alpha\mathbf{Z}_a = (1,\partial_t z_a)\times(0,\partial_\alpha z_a) = (-\partial_t z\cdot\partial_\alpha z^\perp, \partial_\alpha z^\perp)_a,$$

the outward (w.r.t. Ω_{tur}) unit normal vector to z_a is

$$n_a = a\frac{(-\partial_t z\cdot\partial_\alpha z^\perp,\partial_\alpha z^\perp)_a}{|(-\partial_t z\cdot\partial_\alpha z^\perp,\partial_\alpha z^\perp)_a|}.$$

Thus, by splitting $[0,T]\times\mathbb{R}^2$ into each Ω_r for $r=+,-,\mathrm{tur}$ and integrating by parts, we deduce that, for every test function $\Psi\in C_c^1(\mathbb{R}^3;\mathbb{R}^2)$, we have

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^2}(\bar{v}\cdot\partial_t\Psi + \sigma:\nabla\Psi)\,\mathrm{d}x\,\mathrm{d}s - \int_{\mathbb{R}^2}\bar{v}(t)\cdot\Psi(t)\,\mathrm{d}x + \int_{\mathbb{R}^2}v^\circ\cdot\Psi^\circ\,\mathrm{d}x\\ &= \sum_{a=\pm}\int_0^t\int_{\mathbb{T}}((\partial_t z\cdot\partial_\alpha z^\perp)[\bar{v}] - [\sigma]\partial_\alpha z^\perp)_a\cdot\Psi_a\,\mathrm{d}\alpha\,\mathrm{d}s - \int_0^t\int_{\Omega_{\mathrm{tur}}(\alpha)}(\mathrm{div}R)\cdot\Psi\,\mathrm{d}x\,\mathrm{d}s,\end{aligned}$$

where we have applied $\bar{v}|_{t=0}=v^\circ$ and (4.40). Therefore, (4.25) is equivalent to (4.43a) and

$$(\partial_t z\cdot\partial_\alpha z^\perp)_a[\bar{v}]_a = [\sigma]_a\partial_\alpha z_a^\perp \quad \text{on} \quad z_a. \tag{4.45}$$

On the one hand, by (4.31) (recall sec. 1.31.3),

$$(\partial_t z \cdot \partial_\alpha z^\perp)_a [\bar{v}]_a = -\varpi \partial_t z_a \cdot \frac{\partial_\alpha z_a^\perp}{|\partial_\alpha z_a|^2} \partial_\alpha z_a.$$

On the other hand, let us split

$$[\sigma]_a = [\bar{v} \otimes \bar{v}]_a + [\bar{p}]_a + [R]_a.$$

Then, by applying the identity $(a \otimes b)c = a(b \cdot c)$ we get

$$[\bar{v} \otimes \bar{v}]_a \partial_\alpha z_a^\perp = \bar{v}_a^+ (\bar{v}^+ \cdot \partial_\alpha z^\perp)_a - \bar{v}_a^- (\bar{v}^- \cdot \partial_\alpha z^\perp)_a = [\bar{v}]_a (\mathscr{B} \cdot \partial_\alpha z^\perp)_a = -\varpi \mathscr{B}_a \cdot \frac{\partial_\alpha z_a^\perp}{|\partial_\alpha z_a|^2} \partial_\alpha z_a.$$

Therefore, since $[R]_\pm = \mp R_\pm$, (4.45) reads as

$$\pm R_\pm \partial_\alpha z_\pm^\perp = i \left([\bar{p}]_\pm - \tfrac{1}{2} i \varpi (\partial_t z_\pm - \mathscr{B}_\pm) \cdot \frac{\partial_\alpha z_\pm^\perp}{|\partial_\alpha z_\pm|^2} \partial_\alpha z_\pm \right) = i b_\pm,$$

where we applied Proposition 4.41. □

Remark 4.2.1. Observe that in the case of a single sheet ($z_+ = z_-$) the above analysis reduces to the derivation of the Birkhoff-Rott system (4.9) from the weak formulation of the incompressible Euler equation

$$b := \partial_t \tilde{\varpi} \partial_\alpha z - \varpi(\partial_t z - \mathscr{B}) = 0.$$

As a result, Proposition 4.2.2 implies continuity of the pressure along the sheet

$$[p] = b \cdot \frac{\partial_\alpha z}{|\partial_\alpha z|^2} = 0,$$

thus generalizing the observation made in [25] that the continuity of the pressure can be deduced from the weak formulation and need not appear as an assumption as for instance in [96].

In the case of two sheets ($z_+ \neq z_-$) one may set $R = 0$ by imposing $b_+ = b_- = 0$, but this seems as problematic as the single sheet case. In other words, the Reynolds stress R allows to relax the ill-posed equations $b_+ = b_- = 0$ (cf. section 4.5).

Solvability of (4.43)

Next we discuss necessariy and sufficient conditions for solvability of the boundary value problem (4.43). Recall our notation $\langle b \rangle := \frac{1}{2}(b_+ + b_-)$ and $\{b\} := \frac{1}{2}(b_+ - b_-)$ introduced in section 1.3.

We consider the isomorphism $\mathrm{M} : \mathbb{C} \to \mathring{\mathbb{R}}^{2\times 2}_{\mathrm{sym}}$ given by

$$\mathrm{M}(z) = \begin{pmatrix} -z_1 & z_2 \\ z_2 & z_1 \end{pmatrix},$$

which satisfies the identity

$$\mathrm{M}(z) w = -(zw)^*.$$

Note that $|\mathrm{M}(z)| = |z|$ (where $|\cdot|$ denotes the operator norm of the matrix on the l.h.s. and the norm of the complex number on the r.h.s.). Given $\Omega \subset \mathbb{C}$ open, any $f \in C^1(\Omega; \mathbb{R}^2)$ satisfies (Hol $\equiv$ holomorphic functions)

$$f \in \mathrm{Hol}(\Omega) \quad \Leftrightarrow \quad \mathrm{div}(\mathrm{M}(f)) = 0 \quad \text{on} \quad \Omega.$$

Proposition 4.2.4. *The boundary value problem* (4.43) *admits a solution* $R = R(t, x)$ *uniformly bounded in* $t > 0$ *if and only if*

(a) $\langle b \rangle = t\partial_\alpha(q\partial_\alpha z)$

for some $q = q_1 + iq_2$ *satisfying*

(b) $\int(q_1|\partial_\alpha z|^2 - c\{b\} \cdot \tau^\perp) = 0,$

(c) $q_1^{(0)} = c\{b\}^{(0)} \cdot \tau^\perp.$

The question of solvability involves two issues. First, since R is divergence-free, Gauss divergence theorem leads to an integrability condition for the boundary values - this is represented by (a)(b) above. Secondly, the thickness of the domain $\Omega_{\rm tur}(t)$ is order $\sim t$, so that, in order to make sure that R is uniformly bounded in t, there is a necessary matching condition at $t = 0$ - represented by (c) above.

We split the proof of Proposition 4.2.4 in two parts. First we look at necessary and sufficient conditions on b_a for solvability of (4.43) at any fixed time slice. In the following we will use the notation $G := g \circ \mathbf{Z} = g^\sharp$ to denote the change of variables adapted to $\Omega_{\rm tur}$ as

$$G(t, \alpha, a) = g^\sharp(t, \alpha, a) = g(t, z_a(t, \alpha)). \tag{4.46}$$

In the following lemma we fix $t > 0$ and, for ease of notation, suppress dependence on t.

Lemma 4.2.1. *Given* $b_a \in C(\mathbb{T}; \mathbb{R}^2)$, *there exists* $R \in C^1(\Omega_{\rm tur}; \mathbb{R}^{2\times 2}_{sym}) \cap C(\bar{\Omega}_{\rm tur}; \mathbb{R}^{2\times 2}_{sym})$ *solving* (4.43) *if and only if the following compatibility conditions hold*

$$\int \langle b \rangle = 0, \tag{4.47a}$$

$$\int \langle b \cdot z \rangle = 0. \tag{4.47b}$$

In this case,

$$R = \begin{pmatrix} \partial_{22} g & -\partial_{12} g \\ -\partial_{12} g & \partial_{11} g \end{pmatrix} + \mathrm{M}f,$$

where

$$f := \zeta^* K_{x_0} + \eta K'_{x_0} \quad \textit{with} \quad \begin{aligned} \zeta &:= \int \{b\}, \\ \eta &:= \zeta \cdot x_0 - \int \{b \cdot z\}, \end{aligned}$$

for some $g \in C^3(\Omega_{\rm tur}) \cap C^2(\bar{\Omega}_{\rm tur})$ *satisfying*

$$\partial_\alpha(\nabla g)_\pm = \pm b_\pm - (f\partial_\alpha z)^*_\pm. \tag{4.48}$$

Proof. Let us start assuming that R is a solution to (4.43). Then, Gauss divergence theorem implies (4.47a)

$$0 = \int_{\Omega_{\rm tur}} \mathrm{div} R = 2\int \{R\partial_\alpha z^\perp\} = 2i\int \langle b \rangle.$$

But then, for any simple closed curve $\gamma \in C^1(\mathbb{T};\Omega_{\rm tur})$ $\circlearrowleft$-surrounding x_0 we have $\int_\gamma R\,\mathrm{d}y^\perp = i\int\{b\}$. Consequently, there exists $\Psi = (\psi_1,\psi_2) \in C^2(\Omega_{\rm tur};\mathbb{R}^2)\cap C^1(\bar\Omega_{\rm tur};\mathbb{R}^2)$ so that

$$R = \begin{pmatrix} -\partial_2\psi_1 & \partial_1\psi_1 \\ -\partial_2\psi_2 & \partial_1\psi_2 \end{pmatrix} + \mathrm{M}(\zeta^* K_{x_0}). \tag{4.49}$$

Indeed, for any simple closed curve $\gamma \in C^1(\mathbb{T};\Omega_{\rm tur})$ $\circlearrowleft$-surrounding x_0, using $\mathrm{M}(z)w = -(zw)^*$, we have

$$\int_\gamma (R - \mathrm{M}(\zeta^* K_{x_0}))\,\mathrm{d}x^\perp = \int_\gamma R\,\mathrm{d}x^\perp - i\zeta = i\int\{b\} - i\zeta = 0.$$

Therefore Ψ can be recovered via

$$\Psi(x) - \Psi_{x_{\rm tur}} = \int_\gamma R\,\mathrm{d}y^\perp - i\zeta\left(\int_\gamma K_{x_0}\,\mathrm{d}y\right)^*,$$

for any path $\gamma \in C^1([0,1];\Omega_{\rm tur}\setminus L_{x_0})$ with $\gamma(0) = x_{\rm tur}$ and $\gamma(1) = x$, where $\Psi(x_{\rm tur}) = \Psi_{x_{\rm tur}}$ is an arbitrary constant vector.

Now, since R is symmetric, necessarily $\mathrm{div}\Psi = 0$. Hence, since

$$\partial_\alpha\Psi_\pm = ((\nabla\Psi)\partial_\alpha z)_\pm = ((R - \mathrm{M}(\zeta^* K_{x_0}))\partial_\alpha z^\perp)_\pm = i(\pm b - \zeta(K_{x_0}\partial_\alpha z)^*)_\pm,$$

with

$$\int (\zeta^* K_{x_0}\partial_\alpha z)_a^* \cdot z_a = \zeta\cdot\int_{z_a} x K_{x_0}\,\mathrm{d}x = \zeta\cdot x_0,$$

Gauss divergence theorem implies (4.47b)

$$0 = \int_{\Omega_{\rm tur}} \mathrm{div}\Psi = 2\int\{\Psi\cdot\partial_\alpha z^\perp\} = 2\int\{\partial_\alpha\Psi^\perp\cdot z\} = -2\int\langle b\cdot z\rangle.$$

Therefore, there is some $g \in C^3(\Omega_{\rm tur})\cap C^2(\bar\Omega_{\rm tur})$ so that

$$\Psi = \nabla^\perp g + i\eta K_{x_0}^*. \tag{4.50}$$

Indeed, analogously to above, g can be recovered via

$$g(x) - g_{x_{\rm tur}} = \int_\gamma \Psi\cdot\,\mathrm{d}y^\perp - \eta\left(\int_\gamma K_{x_0}\,\mathrm{d}y\right)_1,$$

for any path $\gamma \in C^1([0,1];\Omega_{\rm tur}\setminus L_{x_0})$ with $\gamma(0) = x_{\rm tur}$ and $\gamma(1) = x$, where $g(x_{\rm tur}) = g_{x_{\rm tur}}$ may be chosen and $\eta \in \mathbb{C}$ is given by the circulation to guarantee the continuity of g on $\Omega_{\rm tur}$

$$0 = \int_{z_a} \Psi\cdot\,\mathrm{d}x^\perp - \eta\left(\int_{z_a} K_{x_0}\,\mathrm{d}x\right)_1 = \left(\zeta\cdot x_0 - \int\{b\cdot z\}\right) - \eta.$$

Now, (4.50) and the Cauchy-Riemann equations yield

$$\begin{pmatrix} -\partial_2\psi_1 & \partial_1\psi_1 \\ -\partial_2\psi_2 & \partial_1\psi_2 \end{pmatrix} = \begin{pmatrix} \partial_{22}g & -\partial_{12}g \\ -\partial_{12}g & \partial_{11}g \end{pmatrix} + \mathrm{M}(\eta K_{x_0}').$$

Finally, g must satisfy the boundary conditions

$$\partial_\alpha(\nabla g)_\pm = (\nabla^2 g\partial_\alpha z)_\pm = -i((R-\mathrm{M}f)\partial_\alpha z^\perp)_\pm = \underbrace{\pm b_\pm - (f\partial_\alpha z)^*_\pm}_{\equiv a_\pm}. \tag{4.51}$$

Notice that

$$\int a_\pm = \pm\int b_\pm - \left(\int_{z_\pm} f\,\mathrm{d}x\right)^* = \int\{b\} - \zeta = 0.$$

Then, (4.51) reads as

$$(\nabla g)_\pm = \int_0^\alpha a_\pm\,\mathrm{d}\alpha_1 + o_\pm, \tag{4.52}$$

for some constant vectors $o_a \in \mathbb{R}^2$. Then, since $\nabla G = \nabla z(\nabla g)^\sharp$, (4.52) is equivalent to

$$(\partial_\alpha G)_\pm = \left(\int_0^\alpha a_\pm\,\mathrm{d}\alpha_1 + o_\pm\right)\cdot\partial_\alpha z_\pm, \tag{4.53a}$$

$$(\partial_\alpha G)_\pm = \left(\int_0^\alpha a_\pm\,\mathrm{d}\alpha_1 + tco_\pm\right)\cdot\tau^\perp. \tag{4.53b}$$

But then notice that

$$\begin{aligned}\int\left(\int_0^\alpha a_\pm\,\mathrm{d}\alpha_1 + o_\pm\right)\cdot\partial_\alpha z_\pm = -\int(a\cdot z)_\pm = \left(\int_{z_\pm} xf\,\mathrm{d}x \mp \int(b\cdot z)_\pm\right)_1 \\ = (\alpha\cdot x_0 - \eta) - \int\{b\cdot z\} = 0.\end{aligned}$$

Hence, (4.53a) reads as

$$G_\pm = \int_0^\alpha\left(\int_0^{\alpha_1} a_\pm\,\mathrm{d}\alpha_2 + o_\pm\right)\cdot\partial_\alpha z_\pm\,\mathrm{d}\alpha_1 + d_\pm, \tag{4.54}$$

for some constants $d_a \in \mathbb{R}$.

Conversely, the easiest way to define G in the interior from the boundary conditions (4.54) (4.53b) is by means of the Lagrange interpolation. Since there are four conditions, we consider the Lagrange polynomial of degree 3 on a

$$L(\alpha,\lambda) := \sum_{k=0}^{3} l_k(\alpha)\lambda^k, \tag{4.55}$$

whose coefficients l_k are determined by

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & -2 & 3 \end{pmatrix}\begin{pmatrix} l_0 \\ l_1 \\ l_2 \\ l_3 \end{pmatrix} = \begin{pmatrix} G_+ \\ G_- \\ (\partial_\lambda G)_+ \\ (\partial_\lambda G)_- \end{pmatrix}.$$

The solution of the above linear system is

$$l_0 = \langle G\rangle - \tfrac{1}{2}\{\partial_\lambda G\}, \qquad l_1 = \tfrac{1}{2}(3\{G\} - \langle \partial_\lambda G\rangle),$$
$$l_2 = \tfrac{1}{2}\{\partial_\lambda G\}, \qquad l_3 = \tfrac{1}{2}(\langle \partial_\lambda G\rangle - \{G\}).$$

Thus, any g has the form $g^\sharp = G = L + H$, for some solution H to the homogeneous problem $(\partial_\lambda^k H)_\pm = 0$ for $k = 0, 1$. This concludes the proof. $\square$

Lemma 4.2.1 above shows how $R(t)$ is at each time slice $0 < t \leq T$. Next, we must guarantee that $R(t)$ remains uniformly bounded as $t \to 0$.

Proof of Proposition 4.2.4. Recalling the definition of $G = g^\sharp$ from (4.46) we have

$$\begin{aligned} \partial_\alpha G &= (\nabla g)^\sharp \cdot \partial_\alpha z, \\ \partial_\lambda G &= tc(\nabla g)^\sharp \cdot \tau^\perp, \end{aligned} \tag{4.56}$$

and

$$\partial_{\alpha\alpha} G = \partial_\alpha z \cdot (\nabla^2 g)^\sharp \partial_\alpha z + (\nabla g)^\sharp \cdot \partial_{\alpha\alpha} z, \tag{4.57a}$$
$$\partial_{\alpha\lambda} G = tc\partial_\alpha z \cdot (\nabla^2 g)^\sharp \tau^\perp + t(\nabla g)^\sharp \cdot \partial_\alpha (c\tau^\perp), \tag{4.57b}$$
$$\partial_{\lambda\lambda} G = (tc)^2 \tau^\perp \cdot (\nabla^2 g)^\sharp \tau^\perp. \tag{4.57c}$$

Then, $\nabla^2 g$ (equivalently R) is uniformly bounded if and only if, as $t \to 0$,

$$\partial_\alpha^k \partial_\lambda^n G = O(t^n), \quad 0 \leq k + n \leq 2. \tag{4.58}$$

By considering the Taylor expansion of $\lambda \mapsto G(t, \alpha, \lambda)$ on $[-1, 1]$ we see that (4.58) implies

$$G_+ - G_- = O(t), \quad \partial_\lambda G_+ - \partial_\lambda G_- = O(t^2), \quad (G_+ + \partial_\lambda G_+) - (G_- - \partial_\lambda G_-) = O(t^2),$$

that is,

$$\langle \partial_\lambda^n G\rangle = O(t^n), \quad \{\partial_\lambda^n G\} = O(t^{n+1}), \quad n = 0, 1, \tag{4.59a}$$
$$\{G\} - \langle \partial_\lambda G\rangle = O(t^2). \tag{4.59b}$$

Using (4.54)(4.53b), these terms are

$$\begin{aligned}
\langle G\rangle &= \int_0^\alpha \left(\left(\int_0^{\alpha_1} \langle a\rangle \,\mathrm{d}\alpha_2 + \langle o\rangle \right) \cdot \partial_\alpha z + t \left(\int_0^{\alpha_1} \{a\} \,\mathrm{d}\alpha_2 + \{o\} \right) \cdot \partial_\alpha (c\tau^\perp) \right) \mathrm{d}\alpha_1 + \langle d\rangle, \\
\{G\} &= \int_0^\alpha \left(\left(\int_0^{\alpha_1} \{a\} \,\mathrm{d}\alpha_2 + \{o\} \right) \cdot \partial_\alpha z + t \left(\int_0^{\alpha_1} \langle a\rangle \,\mathrm{d}\alpha_2 + \langle o\rangle \right) \cdot \partial_\alpha (c\tau^\perp) \right) \mathrm{d}\alpha_1 + \{d\}, \\
\langle \partial_\lambda G\rangle &= tc \left(\int_0^\alpha \langle a\rangle \,\mathrm{d}\alpha_1 + \langle o\rangle \right) \cdot \tau^\perp, \\
\{\partial_\lambda G\} &= tc \left(\int_0^\alpha \{a\} \,\mathrm{d}\alpha_1 + \{o\} \right) \cdot \tau^\perp.
\end{aligned}$$

Observe that $\langle \partial_\lambda^n G\rangle = O(t^n)$ for $n = 0, 1$. The conditions $\{\partial_\lambda^n G\} = O(t^{n+1})$ for $n = 0, 1$ implies that $\{a\}^{(0)} = \{o\}^{(0)} = \{d\}^{(0)} = 0$. In particular, since $\langle b\rangle^{(0)} = \{a\}^{(0)} = 0$, the zero-mean condition (4.47a) reads as (a):

$$\langle b\rangle = t\partial_\alpha (q\partial_\alpha z),$$

for some $q(t,\alpha) = q_1(t,\alpha) + iq_2(t,\alpha)$. We may assume w.l.o.g. that

$$(q\partial_\alpha z)|_{\alpha=0} = \{o\}^{(1} + t\fint_{-1}^{1} f_\lambda \,\mathrm{d}\lambda (c\tau^\perp)|_{\alpha=0}. \tag{4.60}$$

By (a), the zero-mean condition (4.47b) reads as (b):

$$0 = \int \langle b\cdot z\rangle = \int (\langle b\rangle \cdot z + tc\{b\}\cdot\tau^\perp) = t\int (c\{b\}\cdot\tau^\perp - q_1|\partial_\alpha z|^2).$$

Coming back to (4.59b), an integration by parts yields

$$t\int_0^\alpha \left(\int_0^{\alpha_1} \langle a\rangle \,\mathrm{d}\alpha_2 + \langle o\rangle\right)\cdot \partial_\alpha (c\tau^\perp)\,\mathrm{d}\alpha_1 = \langle \partial_\lambda G\rangle - tc(0)\langle o\rangle \cdot \tau(0)^\perp - t\int_0^\alpha c\langle a\rangle\cdot\tau^\perp \,\mathrm{d}\alpha_1,$$

from which we deduce that

$$\begin{aligned} &\{G\} - \langle\partial_\lambda G\rangle \\ &= t\left(\int_0^\alpha \left(\left(\int_0^{\alpha_1} \{a\}^{(1}\,\mathrm{d}\alpha_2 + \{o\}^{(1}\right)\cdot\partial_\alpha z - c\langle a\rangle\cdot\tau^\perp\right)\mathrm{d}\alpha_1 + \underbrace{\{d\}^{(1} - c(0)\langle o\rangle\cdot\tau(0)^\perp}_{\equiv\tilde{d}}\right). \end{aligned} \tag{4.61}$$

In particular, (4.59b) implies that $\tilde{d}^{(0)} = 0$. Let us split (4.61) in terms of b_a and f. On the one hand,

$$c\langle a\rangle\cdot\tau^\perp = c\{b\}\cdot\tau^\perp - (\langle f\partial_\alpha z\rangle c\tau^\perp)_1.$$

On the other hand, since $f(t)\in \mathrm{Hol}(\mathbb{C}\setminus\{x_0\})$, Cauchy's integral theorem implies

$$\int_0^{\alpha_1} \{f\partial_\alpha z\}\,\mathrm{d}\alpha_2 = \fint_{-1}^{1} f_\lambda \,\mathrm{d}\lambda tc\tau^\perp\Big|_0^{\alpha_1},$$

and, by (4.60),

$$\begin{aligned} \int_0^{\alpha_1} \{a\}^{(1}\,\mathrm{d}\alpha_2 + \{o\}^{(1} &= \int_0^{\alpha_1} \langle b\rangle^{(1}\,\mathrm{d}\alpha_2 + \{o\}^{(1} - \left(\fint_{-1}^{1} f_\lambda\,\mathrm{d}\lambda c\tau^\perp\Big|_0^{\alpha_1}\right)^* \\ &= q\partial_\alpha z - \left(\fint_{-1}^{1} f_\lambda \,\mathrm{d}\lambda c\tau^\perp\right)^*. \end{aligned} \tag{4.62}$$

Therefore, (4.61) reads as

$$\{G\} - \langle\partial_\lambda G\rangle = t\int_0^\alpha \left(q_1|\partial_\alpha z|^2 - c\{b\}\cdot\tau^\perp + t\tilde{d}^{(1} + (Fc\tau^\perp)_1\right)\mathrm{d}\alpha_1, \tag{4.63}$$

where we have abbreviated

$$F \equiv \langle f\partial_\alpha z\rangle - \fint_{-1}^{1} f_\lambda \,\mathrm{d}\lambda \partial_\alpha z.$$

It is straightforward to check that $F^{(0)} = F^{(1)} = 0$, so $F = t^2F^{(2}$. Therefore, the condition (4.59b) requires $q_1^{(0)} = c\{b\}^{(0)}\cdot\tau^\perp$, i.e. (c).

Conversely, the Lagrange polynomial L given in (4.55) satisfies (4.58) if and only if

$$\partial_\alpha^k l_n,\ldots,\partial_\alpha^k l_3 = O(t^n), \quad 0\le k+n\le 2,$$

which is indeed equivalent to (4.59). □

4.3 The dissipation

Recall from Definition 4.2.1 and Theorem 6.2.1 that our weak solutions (v, p) to IE are obtained from a subsolution $(\bar{v}, \bar{p}, R)$ via convex integration, and satisfy

$$\tfrac{1}{2}|v|^2 = e, \qquad e + p = \tfrac{1}{2}(|\bar{v}|^2 + \mathrm{tr}R) + \bar{p}, \tag{4.64}$$

where

$$e := \tfrac{1}{2}|\bar{v}|^2 + |\mathring{R}| + e', \tag{4.65}$$

for some error function e' strictly positive on Ω_{tur} while vanishing outside. This allows us to calculate the associated dissipation measure:

Proposition 4.3.1. *Let $(\bar{v}, \bar{p}, R)$ be a strict subsolution w.r.t. some $e \in C^0(\Omega_{\mathrm{tur}}; \mathbb{R}_+)$ and assume that it is C^1 outside $z_+ \cup z_- = \partial\Omega_{\mathrm{tur}}$ with $z_\pm$ parametrized by C^1 curves $z_\pm = z_\pm(t, \alpha)$. Then the dissipation measure $D(t)$ from Definition 4.1.2 is supported in Ω_{tur} with*

$$\begin{aligned}\langle D(t), \psi\rangle = &\sum_{a=\pm} \int_0^t \int ([e]_a \partial_t z_a + [(e+p)\bar{v}]_a) \cdot \partial_\alpha z_a^\perp \psi_a \,\mathrm{d}\alpha\,\mathrm{d}s \\ &- \int_0^t \int_{\Omega_{\mathrm{tur}}(s)} \Big(\partial_t(e - \tfrac{1}{2}|\bar{v}|^2) - \bar{v}\cdot \mathrm{div}\mathring{R} + (v - \bar{v})\cdot\nabla(e+p)\Big)\psi \,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{4.66}$$

Proof. Since $(\bar{v}, \bar{p}, R)$ is piecewise C^1 outside $\partial\Omega_{\mathrm{tur}}$, by multiplying the (relaxed) momentum balance equation (4.24a) by $\bar{v}$ we get

$$\tfrac{1}{2}\partial_t|\bar{v}|^2 + \mathrm{div}((\tfrac{1}{2}|\bar{v}|^2 + \bar{p})\bar{v}) + \bar{v}\cdot\mathrm{div}R = 0 \quad \text{outside } z_+ \cup z_-,$$

or, by (4.64), equivalently

$$\tfrac{1}{2}\partial_t|\bar{v}|^2 + \mathrm{div}((e+p)\bar{v}) + \bar{v}\cdot\mathrm{div}\mathring{R} = 0 \quad \text{outside } z_+ \cup z_-. \tag{4.67}$$

Firstly, by adding and subtracting $\bar{v}$, we split the dissipation into

$$\langle D(t), \psi\rangle = \int_0^t \int_{\mathbb{R}^2} (e\partial_t\psi + (e+p)\bar{v}\cdot\nabla\psi)\,\mathrm{d}x\,\mathrm{d}s - \int_{\mathbb{R}^2} e\psi\,\mathrm{d}x\Big|_{s=0}^{s=t} \tag{4.68a}$$

$$+ \int_0^t \int_{\mathbb{R}^2} (e+p)(v - \bar{v})\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}s, \tag{4.68b}$$

where the first term (4.68a) only depends on the subsolution by (4.64)(4.65), while the second term (4.68b) is the corresponding fluctuation. On the one hand, similarly to the proof of Proposition 4.2.3, an integrating by parts yields

$$(4.68a) = \sum_{a=\pm} \int_0^t \int ([e]_a \partial_t z_a + [(e+p)\bar{v}]_a) \cdot \partial_\alpha z_a^\perp \psi_a \,\mathrm{d}\alpha\,\mathrm{d}s \tag{4.69a}$$

$$- \int_0^t \int_{\Omega_{\mathrm{tur}}(s)} (\partial_t e + \mathrm{div}((e+p)\bar{v}))\psi\,\mathrm{d}x\,\mathrm{d}s, \tag{4.69b}$$

where we have applied that $(\bar{v}, \bar{p}, R) = (v, p, 0)$ outside Ω_{tur} and (4.67). Indeed, by (4.67), the term (4.69b) reads as

$$(4.69\text{b}) = -\int_0^t \int_{\Omega_{\text{tur}}(s)} (\partial_t(e - \tfrac{1}{2}|\bar{v}|^2) - \bar{v} \cdot \operatorname{div}\mathring{R})\psi \, \mathrm{d}x \, \mathrm{d}s.$$

On the other hand, using that $v, \bar{v} \in L^\infty_{\text{div}}$, an integration by parts yields

$$(4.68\text{b}) = -\int_0^t \int_{\Omega_{\text{tur}}(s)} (v - \bar{v}) \cdot \nabla(e + p)\psi \, \mathrm{d}x \, \mathrm{d}s.$$

This concludes the proof. □

Note that there is some ambiguity in Definition 4.2.1 because the trace part of R may be absorbed into the pressure; in particular we have

$$R + \bar{p}\ \mathrm{Id} = \bar{R} + p\ \mathrm{Id},$$

where $\bar{R} = \mathring{R} + (e - \frac{1}{2}|\bar{v}|^2)\mathrm{Id}$ (cf. formulas (4.64)(4.65)). Nevertheless, the expression (4.66), which does not depend on the specific choice of $\bar{p}$ and $\operatorname{tr}R$, is well defined.

Our aim now is to calculate the initial dissipation measure in terms of $(^\circ, \varpi^\circ)$. In particular, since $e > \frac{1}{2}|\bar{v}|^2 + |\mathring{R}|$, having defined $(\bar{v}, \bar{p})$ by the *ansatz* (4.28) and Bernoulli's law (4.39) our next aim is to minimize $|\mathring{R}|$ at time $t = 0$ among all solutions of the boundary value problem (4.43). Recall our notation $R^\sharp(t, \alpha, \lambda) = R(t, z_\lambda(t, \alpha))$.

Proposition 4.3.2. *In general, for any uniformly bounded solution R of* (4.43),

$$|\mathring{R}^{\sharp(0)}| \geq |\{b\}^{(0)} \cdot \partial_\alpha z^{\circ\perp}|.$$

Equality is attained if and only if $\operatorname{tr}R^{\sharp(0)} = 2\{b\}^{(0)} \cdot \partial_\alpha z^\circ$, which can be achieved if, in the setting of Proposition 4.2.4, we have in addition

(d) $q_2^{(0)} = c\{b\}^{(0)} \cdot \partial_\alpha z^\circ$,

(e) $\left(q_1|\partial_\alpha z|^2 - c\{b\} \cdot \tau^\perp\right)^{(1)} = 0.$

Proof. Step 1. First of all we claim that $|\mathring{R}^\sharp|$ at time $t = 0$ is given by the formula

$$|\mathring{R}^{\sharp(0)}| = |\{b\}^{(0)} - \tfrac{1}{2}\operatorname{tr}R^{\sharp(0)}\partial_\alpha z^\circ|. \tag{4.70}$$

Recalling Proposition 4.2.1 let us decompose R as

$$\mathring{R} = \mathrm{M}(\tfrac{1}{2}(\partial_{11} - \partial_{22})g - i\partial_{12}g + f), \qquad \operatorname{tr}R = \Delta g.$$

Therefore $|\mathring{R}| = |\frac{1}{2}(\partial_{11} - \partial_{22})g + i\partial_{12}g + f^*|$. In particular, since

$$(\tfrac{1}{2}(\partial_{11} - \partial_{22})g + i\partial_{12}g)^\sharp \partial_\alpha z^* = \partial_\alpha(\nabla g)^\sharp - \tfrac{1}{2}(\Delta g)^\sharp \partial_\alpha z,$$

we have

$$|\mathring{R}^\sharp||\partial_\alpha z| = |\partial_\alpha(\nabla g)^\sharp + (f^\sharp \partial_\alpha z)^* - \tfrac{1}{2}\operatorname{tr}R^\sharp \partial_\alpha z|. \tag{4.71}$$

Solving the linear system (4.56) and applying $\partial_\lambda G^{(0)} = 0$ we obtain

$$(\nabla g)^\sharp = \frac{1}{\partial_\alpha z \cdot \partial_\alpha z^\circ}\left(\partial_\alpha G \partial_\alpha z^\circ + c^{-1}(\partial_\lambda G)^{(1}\partial_\alpha z^\perp\right). \tag{4.72}$$

Hence, recalling the decomposition $G = L + H$ in (4.55) and the proof of Proposition 4.2.4, we deduce

$$\begin{aligned}
(\nabla g)^{\sharp(0)} &= \left((\partial_\alpha G)^{(0)} + ic^{-1}(\partial_\lambda G)^{(1)}\right)\partial_\alpha z^\circ \\
&= \left(\partial_\alpha l_0^{(0)} + ic^{-1} l_1^{(1)}\right)\partial_\alpha z^\circ \\
&= \left(\partial_\alpha \langle G\rangle^{(0)} + ic^{-1}\langle \partial_\lambda G\rangle^{(1)}\right)\partial_\alpha z^\circ \\
&= \int_0^\alpha \langle a\rangle^{(0)}\,\mathrm{d}\alpha_1 + \langle o\rangle^{(0)}.
\end{aligned}$$

Therefore, (4.70) follows from

$$(\partial_\alpha(\nabla g)^\sharp + (f^\sharp\partial_\alpha z)^*)^{(0)} = \{b\}^{(0)}.$$

Step 2. Next, we derive the formula

$$\mathrm{tr}R^{\sharp(0)} = \{b\}^{(0)}\cdot\partial_\alpha z^\circ + c^{-1}q_2^{(0)} + \tfrac{1}{2}c^{-2}(6l_3\lambda + \partial_\lambda^2 H)^{(2)}.$$

Let us consider the equivalent matrix $S = S(t,\alpha,\lambda)$

$$S := Q(\nabla^2 g)^\sharp Q, \tag{4.73}$$

given by the orthogonal change of basis $Q := \mathrm{M}(-(\partial_\alpha z^\circ)^*)$ ($Q^2 = \mathrm{Id}$), that is,

$$(\nabla^2 g)^\sharp = QSQ \quad \text{and} \quad \mathrm{tr}R = \mathrm{tr}S.$$

Thus, using $Q\partial_\alpha z^\circ = (1,0)$, $Q\partial_\alpha z^{\circ\perp} = (0,1)$, (4.57) yields

$$\begin{aligned}
(\partial_\alpha^2 G)^{(0)} &= s_{11}^{(0)} + (\nabla g)^{\sharp(0)}\cdot\partial_\alpha^2 z^\circ, \\
\tfrac{1}{2}(\partial_\lambda^2 G)^{(2)} &= c^2 s_{22}^{(0)}.
\end{aligned}$$

Hence, since

$$(\partial_\alpha^2 G)^{(0)} = \partial_\alpha((\nabla g)^{\sharp(0)}\cdot\partial_\alpha z^\circ) = \langle a\rangle^{(0)}\cdot\partial_\alpha z^\circ + (\nabla g)^{\sharp(0)}\cdot\partial_\alpha^2 z^\circ,$$

we have

$$\mathrm{tr}S^{(0)} = \langle a\rangle^{(0)}\cdot\partial_\alpha z^\circ + c^{-2}(\partial_\lambda^2 G)^{(2)},$$

with $G = L + H$ and

$$(\partial_\lambda^2 L)^{(2)} = 2l_2^{(2)} + 6l_3^{(2)}\lambda.$$

On the one hand, by (4.62) we get

$$2l_2^{(2)} = \{\partial_\lambda G\}^{(2)} = c\left(\int_0^\alpha \{a\}^{(1)}\,\mathrm{d}\alpha_1 + \{o\}^{(1)}\right)\cdot\tau^\perp = cq_2^{(0)} - c^2(f^{(0)}\partial_\alpha z^\circ)^*\cdot\partial_\alpha z^\circ,$$

with $(f^{(0)}\partial_\alpha z^\circ)^* = (\{b\} - \langle a\rangle)^{(0)}$. On the other hand, by (4.63) we get

$$2l_3^{(2)} = (\langle\partial_\lambda G\rangle - \{G\})^{(2)} = \int_0^\alpha (c\{b\}\cdot\tau^\perp - q_1|\partial_\alpha z|^2 - t\tilde{d})^{(1)}\,\mathrm{d}\alpha_1.$$

Recalling the average condition (b) in Proposition 4.2.4, then $l_3^{(2)} = 0$ if and only if $\tilde{d}^{(1)} = 0$ and the condition (e) above holds. This concludes the proof by taking $H = 0$. □

Proposition 4.3.3. *Let us assume that there is* (z,ϖ) *satisfying the conditions* (a)-(e) *stated in Propositions 4.2.4 and 4.3.2. Then, the dissipation measure* (4.66) *satisfies* $D \in \mathcal{M}_c([0,T]\times\mathbb{R}^2)$ *with* $\operatorname{supp} D \subset \bar{\Omega}_{\mathrm{tur}}$. *Moreover, given* $0 < \varepsilon \leq \ell_\circ$ *and* $N = 1$, (4.22) *holds.*

Proof. Recalling Proposition 4.3.1 we have

$$\left\langle \frac{D(t_2)-D(t_1)}{t_2-t_1}, \psi \right\rangle = \sum_{a=\pm} \fint_{t_1}^{t_2} \int ([e]_a \partial_t z_a + [(e+p)\bar{v}]_a)\cdot\partial_\alpha z_a^\perp \psi_a \,\mathrm{d}\alpha\,\mathrm{d}s \tag{4.74a}$$

$$- \fint_{t_1}^{t_2} \int_{\Omega_{\mathrm{tur}}(s)} \left(\partial_t(e - \tfrac{1}{2}|\bar{v}|^2) - \bar{v}\cdot\operatorname{div}\mathring{R}\right)\psi\,\mathrm{d}x\,\mathrm{d}s \tag{4.74b}$$

$$- \fint_{t_1}^{t_2} \int_{\Omega_{\mathrm{tur}}(s)} (v-\bar{v})\cdot\nabla(e+p)\psi\,\mathrm{d}x\,\mathrm{d}s. \tag{4.74c}$$

Concerning (4.74b), by applying $e - \frac{1}{2}|\bar{v}|^2 = |\mathring{R}| + e'$ (4.65) and $\operatorname{div} R = 0$ (4.43a), we deduce

$$(4.74\mathrm{b}) = -\fint_{t_1}^{t_2} \int_{\Omega_{\mathrm{tur}}(s)} \left(\partial_t(|\mathring{R}| + e') + \tfrac{1}{2}\bar{v}\cdot\nabla\operatorname{tr}R\right)\psi\,\mathrm{d}x\,\mathrm{d}s.$$

Hence, by Lemma 4.3.3 below and imposing $\partial_t e' \in L^\infty(\Omega_{\mathrm{tur}})$,

$$(4.74\mathrm{b}) \leq O(t_2)\|\psi\|_{L^\infty}.$$

Concerning (4.74c), we can guarantee that (see (4.27)), for any fixed $0 < \varepsilon \leq \ell_\circ$ and time-error function $\mathcal{T} \in C(]0,T];]0,1])$, these weak solutions satisfy

$$\left| \int_{\mathbb{R}^2} (v-\bar{v})\cdot\nabla(e+p)\psi_I\,\mathrm{d}x \right| \leq \mathcal{T}(t),$$

for every interval $I \subset \mathbb{T}$ with $|I| \geq \varepsilon$ and $t \in]0,T]$.

Concerning (4.74a), notice that

$$[(e+p)\bar{v}]_a\cdot\partial_\alpha z_a^\perp = (e+p)_a^+\bar{v}_a^+\cdot\partial_\alpha z_a^\perp - (e+p)_a^-\bar{v}_a^-\cdot\partial_\alpha z_a^\perp = [e+p]_a\mathscr{B}_a\cdot\partial_\alpha z_a^\perp.$$

Hence, since (recall (4.38)-(4.42))

$$\begin{aligned}
[e]_a &= -\tfrac{1}{2}\varpi\frac{\mathscr{B}_a\cdot\partial_\alpha z_a}{|\partial_\alpha z_a|^2} - a(|\mathring{R}| + e')_a,\\
[e+p]_a &= -[\partial_t\phi]_a - \tfrac{a}{2}\operatorname{tr}R_a,
\end{aligned}$$

we deduce

$$(4.74\mathrm{a}) = -\sum_{a=\pm} a \fint_{t_1}^{t_2} \int_{\mathbb{T}} (\mathscr{B}\cdot\mathring{R}\partial_\alpha z^\perp + (|\mathring{R}| + e')\partial_t z\cdot\partial_\alpha z^\perp)_a \psi_a\,\mathrm{d}\alpha\,\mathrm{d}s. \tag{4.75}$$

On the one hand, as we shall in Corollary 4.4.2 below,

$$\mathscr{B}_a^{(0)} = \mathscr{B}^\circ - \tfrac{a}{4}\varpi^\circ\partial_\alpha z^\circ, \tag{4.76}$$

with $\mathscr{B}^\circ$ given in (2.14). Hence

$$\langle\mathscr{B}\rangle^{(0)} = \mathscr{B}^\circ, \qquad \{\mathscr{B}\}^{(0)} = -\tfrac{1}{4}\varpi^\circ\partial_\alpha\mathscr{B}^\circ, \tag{4.77}$$

and so

$$\pm\, b_\pm^{(0)} = \{b\}^{(0)} = -\tfrac{1}{2}\varpi^\circ(c\tau^\perp - \{\mathscr{B}\}^{(0)}) = -\tfrac{1}{2}\varpi^\circ\left(ic + \tfrac{1}{4}\varpi^\circ\right)\partial_\alpha z^\circ. \tag{4.78}$$

Then, by splitting $\mathring{R} = R - \frac{1}{2}(\mathrm{tr}R)\mathrm{Id}$ and applying Proposition 4.3.2, we get

$$(\mathring{R}\partial_\alpha z^\perp)_a^{(0)} = i(\{b\}^{(0)} - \{b\}^{(0)}\cdot\partial_\alpha z^\circ\partial_\alpha z^\circ) = -\{b\}^{(0)}\cdot\partial_\alpha z^{\circ\perp}\partial_\alpha z^\circ = \tfrac{1}{2}c\varpi^\circ\partial_\alpha z^\circ.$$

On the other hand, as we shall see in section 4.5, Proposition 4.2.4 requires $(\partial_t z\cdot\partial_\alpha z^\perp)^{(0)} = \mathscr{B}^\circ\cdot\partial_\alpha z^{\circ\perp}$. Thus, by imposing $e' = O(t)$, we obtain

$$\begin{aligned}
&(\mathscr{B}\cdot\mathring{R}\partial_\alpha z^\perp + (|\mathring{R}| + e')\partial_t z\cdot\partial_\alpha z^\perp)_a^{(0)} \\
&= \left(\mathscr{B}^\circ - \tfrac{a}{4}\varpi^\circ\partial_\alpha z^\circ\right)\cdot\left(\tfrac{1}{2}c\varpi^\circ\partial_\alpha z^\circ\right) + \tfrac{1}{2}c|\varpi^\circ|(\mathscr{B}^\circ\cdot\partial_\alpha z^{\circ\perp} + ac) \\
&= \tfrac{1}{2}c|\varpi^\circ|\left(a\left(c - \tfrac{1}{4}|\varpi^\circ|\right) + B^\circ\right),
\end{aligned}$$

where $B^\circ := \mathscr{B}^\circ\cdot((\mathrm{sgn}\varpi^\circ + i)\partial_\alpha z^\circ)$. This concludes the proof. □

Lemma 4.3.1. *The functional $W_{\mathbb{T}}^{(N)}$ has a global maximum at $c_{\max}^{(N)} = \frac{1}{2}\bar{c}_N|\varpi^\circ|$.*

Proof. Notice that $W \equiv W_{\mathbb{T}}^{(N)}$ satisfies

$$W(c+\psi) - W(c) = dW_c(\psi) - \bar{a}_N\int|\varpi^\circ|\psi^2,$$

where dW_c is the Fréchet derivative of W at c

$$dW_c(\psi) = \bar{a}_N\int|\varpi^\circ|(\bar{c}_N|\varpi^\circ| - 2c)\psi.$$

This concludes the proof. □

Lemma 4.3.2. *Let c given in* (6.60)*. Then, there is $\epsilon(\varpi^\circ,\varepsilon,s) > 0$ so that*

$$W_I(c) \geq \frac{1}{2}s(1-s)\bar{a}_N\bar{c}_N^2\fint_I|\varpi^\circ|^3\,\mathrm{d}\alpha,$$

for every interval $I\subset\mathbb{T}$ with $|I|\geq\varepsilon$.

Proof. Let $c = s\bar{c}_N f_\epsilon$ where we have abbreviated $f\equiv|\varpi^\circ|$ and $f_\epsilon\equiv f*\eta_\epsilon$. We split

$$\begin{aligned}
&\fint_I c|\varpi^\circ|(\bar{c}_N|\varpi^\circ| - c)\,\mathrm{d}\alpha = \frac{s\bar{c}_N^2}{|I|}\int_I f_\epsilon f(f - sf_\epsilon)\,\mathrm{d}\alpha \\
&= \frac{s\bar{c}_N^2}{|I|}\left((1-s)\int_I f^3\,\mathrm{d}\alpha + (1-s)\int_I(f_\epsilon - f)f^2\,\mathrm{d}\alpha + s\int_I f_\epsilon f(f - f_\epsilon)\,\mathrm{d}\alpha\right) \\
&\geq \frac{s\bar{c}_N^2}{|I|}\left((1-s)\int_I f^3\,\mathrm{d}\alpha - \|f\|_{L^\infty(I)}^2\|f_\epsilon - f\|_{L^1(I)}\right).
\end{aligned}$$

Finally, since

$$\alpha_0 \in \mathbb{T} \mapsto \int_{[\alpha_0,\alpha_0+\varepsilon]} f^3 \,\mathrm{d}\alpha > 0$$

is continuous, the lemma holds. □

Lemma 4.3.3. *The $L^\infty(\Omega_{\mathrm{tur}})$-norm of $\nabla_{(t,x)}\mathrm{tr}R$, $\nabla_{(t,x)}|\mathring{R}|$ and $\nabla(e+p)$ are controlled by $\|z\|_{C^{2,\alpha}}$ and $\|\varpi\|_{C^{1,\alpha}}$.*

Proof. Concerning the first two terms $h = \mathrm{tr}R$ and $|\mathring{R}|$, since $\partial_\lambda h^{\sharp(0)} = 0$ by Proposition 4.3.2, we apply (4.72) for h (instead of g) to deduce that ∇h is bounded.
Hence, since $\partial_t h^\sharp = (\partial_t h)^\sharp + (\nabla h)^\sharp \cdot \partial_t z$ and, by (4.71),

$$\partial_t|\mathring{R}|^\sharp = \partial_t\left(\frac{|\partial_\alpha(\nabla g)^\sharp + (f^\sharp\partial_\alpha z)^* - \frac{1}{2}\mathrm{tr}R^\sharp\partial_\alpha z|}{|\partial_\alpha z|}\right),$$

the statement follows by using the (a.e.) inequality $|\partial_t|F|| \leq |\partial_t F|$.

For $\nabla(e+p)$, recalling (4.37)(4.39) and (4.64), it is enough to control

$$\nabla(\tfrac{1}{2}|\bar{v}|^2 + \bar{p})(t,x)^* = \frac{1}{2}\sum_{b=\pm}\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{(\partial_t\tilde{\varpi}\partial_\alpha z_b - \varpi\partial_t z_b)'}{(x-z_b')^2}\,\mathrm{d}\beta, \qquad x \in \Omega_{\mathrm{tur}}(t).$$

Let us abbreviate here $f_b \equiv (\partial_t\tilde{\varpi}\partial_\alpha z_b - \varpi\partial_t z_b)$. Then, the Cauchy's integral formula yields

$$\int_{\mathbb{T}}\frac{f_b'}{(x-z_b')^2}\,\mathrm{d}\beta = \frac{f_b}{\partial_\alpha z_b}\int_{\mathbb{T}}\frac{\partial_\alpha z_b - \partial_\alpha z_b'}{(x-z_b')^2}\,\mathrm{d}\beta - \int_{\mathbb{T}}\frac{f_b - f_b'}{(x-z_b')^2}\,\mathrm{d}\beta.$$

By adding and subtracting $\beta\partial_\alpha^2 z_b$ and also $\beta\partial_\alpha f_b$ we gain integrability (cf. Lemma 4.4.1 below), while the remainder integral

$$\int_{\mathbb{T}}\frac{\beta}{(x-z_b')^2}\,\mathrm{d}\beta$$

can be bounded easily by writing $x = z_a(t,\alpha)$ and applying Lemma 2.2.1. □

4.4 Analysis of the Birkhoff-Rott operator

In light of Proposition 4.3.3, there exists a subsolution adapted to Ω_{tur} provided (z,ϖ) satisfies the conditions (a)-(e) stated in Propositions 4.2.4 and 4.3.2. In section 4.5 we construct such a pair (z,ϖ) by setting its Taylor polynomial $(z,\varpi)^{n_0)}$ satisfying the pointwise conditions (c)-(e), and the corresponding remainder $(z,\varpi)^{(n_0+1}$ satisfying the average conditions (a)-(b). This requires to check that the Taylor decompositions $\mathscr{B}_a = \mathscr{B}_a^{n_0-1)} + t^{n_0}\mathscr{B}_a^{(n_0}$ of the Birkhoff-Rott operators $\mathscr{B}_a$ (4.32) are well defined. This is the goal of this section. Although $n_0 = 3$ in section 4.5, let us keep it general here.

Remark 4.4.1. Recall that $\mathbb{T} = \mathbb{R}/\ell_\circ\mathbb{Z}$. Then, w.l.o.g. in the following we will use the parameter range $\beta \in [-\ell_\circ/2, \ell_\circ/2]$ in (4.80). For ease of notation let us assume that $\ell_\circ = 1$.

Geometric setup

Since z° is closed and C^1, it is simple and regular (4.3a) if and only is chord-arc

$$\mathcal{C}(z^\circ) := \sup_{\alpha,\beta} |\Delta_\beta z^\circ(\alpha)|^{-1} < \infty. \tag{4.79}$$

In particular, $\mathcal{C}(z^\circ) \geq |\partial_\alpha z^\circ|^{-1} = 1$ by the arc-length parametrization.

The chord-arc condition (4.79) is usually imposed when considering Birkhoff-Rott type operators (cf. [135, sec. 1.1]) because it avoids self-intersections (simple) and bad parametrizations (regular). Moreover, it gives a lower bound of the proximity of different points at z°: $|z^\circ(\alpha-\beta) - z^\circ(\alpha)| \geq |\beta|/\mathcal{C}(z^\circ)$ for $|\beta| \leq \ell_\circ/2$, thus measuring the singularity in $\mathscr{B}_a$ at time $t=0$. For $t>0$, $\mathscr{B}_a$ requires to compare different points at the boundary of the turbulence zone. We recall Lemma 2.1.1.

Lemma 4.4.1. *There exists $T_0(\|z\|_{1,\alpha}, \|c\tau\|_{1,\alpha}, \mathcal{C}(z^\circ)) > 0$ such that, for all $0 \leq t \leq T_0$, the following "equi chord-arc" condition holds:*

$$|z_\lambda(t,\alpha) - z_\mu(t,\alpha-\beta)| \geq \frac{1}{4}\sqrt{\frac{\beta^2}{\mathcal{C}(z^\circ)^2} + ((\lambda-\mu)tc(\alpha))^2},$$

for every $\alpha, \beta \in \mathbb{T}$ and $\lambda, \mu \in [-1,1]$.

Analogously to [70, sec. 3], we consider the weighted Hilbert transform on $C^{0,\delta}(\mathbb{T})$:

$$H_\Phi f(\alpha) := \frac{1}{2\pi i}\int_{\mathbb{T}} \Delta_\beta f(\alpha)\Phi(\alpha,\beta)\,\mathrm{d}\beta, \tag{4.80}$$

for the space of weights $L^\infty(\mathbb{T}; C^{k,\delta}(\mathbb{T}))$ with the norm

$$\|\Phi\|_{k,\delta} := \operatorname*{ess\,sup}_{\beta\in\mathbb{T}} \|\Phi(\cdot,\beta)\|_{k,\delta}.$$

We start with the following simple result for boundedness of H_Φ on Hölder spaces, following [70]. We remark that the operator H_Φ is indeed bounded for $\delta' = \delta$ by imposing additional conditions on Φ (see [70, sec. 3]) but we have presented this version for simplicity.

Theorem 4.4.1. *For all $0 \leq \delta' < \delta$ and $k \in \mathbb{N}_0$ there exists $C > 0$ such that*

$$\|H_\Phi f\|_{k,\delta'} \leq \frac{C}{\delta-\delta'}\|\Phi\|_{k,\delta'}\|f\|_{k,\delta},$$

for every $\Phi \in L^\infty(\mathbb{T}; C^{k,\delta'}(\mathbb{T}))$ and $f \in C^{k,\delta}(\mathbb{T})$.

Proof. Let us start with $k=0$. On the one hand,

$$\|H_\Phi f\|_{L^\infty} \leq \frac{2^{-\delta}}{\pi\delta}\operatorname*{ess\,sup}_{\beta\in\mathbb{T}}\|\Phi(\cdot,\beta)\|_{L^\infty}|f|_{C^\delta}.$$

On the other hand, we split

$$\begin{aligned} H_\Phi f(\alpha) - H_\Phi f(\alpha') &= \frac{1}{2\pi i}\int_{\mathbb{T}} \Delta_\beta f(\alpha)\left(\Phi(\alpha,\beta) - \Phi(\alpha',\beta)\right)\mathrm{d}\beta && =: I \\ &+ \frac{1}{2\pi i}\int_{\mathbb{T}} \left(\Delta_\beta f(\alpha) - \Delta_\beta f(\alpha')\right)\Phi(\alpha',\beta)\,\mathrm{d}\beta && =: J. \end{aligned}$$

For I, simply

$$|I| \leq \frac{2^{-\delta}}{\pi\delta} \operatorname*{ess\,sup}_{\beta\in\mathbb{T}} |\Phi(\cdot,\beta)|_{C^{\delta'}} |f|_{C^\delta} |\alpha-\alpha'|^{\delta'}.$$

For J, consider $\gamma = \frac{\delta'}{\delta} \in (0,1)$. By applying $|a+b|^\gamma \leq 2^\gamma(|a|^\gamma + |b|^\gamma)$ we obtain

$$\begin{aligned}
&\left|\Delta_\beta f(\alpha) - \Delta_\beta f(\alpha')\right| = \left|\Delta_\beta f(\alpha) - \Delta_\beta f(\alpha')\right|^{1-\gamma} \left|\Delta_\beta f(\alpha) - \Delta_\beta f(\alpha')\right|^{\gamma} \\
&\leq 2^{1-\gamma}\left(|\Delta_\beta f(\alpha)|^{1-\gamma} + \left|\Delta_\beta f(\alpha')\right|^{1-\gamma}\right) \\
&\quad \cdot 2^\gamma |\beta|^{-\gamma}\left(\left|f(\alpha) - f(\alpha')\right|^\gamma + \left|f(\alpha-\beta) - f(\alpha'-\beta)\right|^\gamma\right) \\
&\leq 8|f|_{C^\delta}|\beta|^{(1-\gamma)\delta-1}|\alpha-\alpha'|^{\gamma\delta},
\end{aligned}$$

and consequently

$$|J| \leq \frac{2^{3-(1-\gamma)\delta}}{\pi(1-\gamma)\delta} \operatorname*{ess\,sup}_{\beta\in\mathbb{T}} |\Phi(\cdot,\beta)|_{L^\infty} |f|_{C^\delta} |\alpha-\alpha'|^{\gamma\delta}.$$

For $k \geq 1$ the result follows by applying the Leibniz rule. □

In particular, we will focus on the following kernels

$$\begin{aligned}
\Phi_{a,b}(t,\alpha,\beta) &:= \frac{\beta}{z_a(t,\alpha) - z_b(t,\alpha-\beta)}, \\
\Phi_{a,b,n_0}(t,\alpha,\beta) &:= \frac{\beta}{(z_a(t,\alpha) - z_b(t,\alpha-\beta))^{n_0-1)}},
\end{aligned}$$

with $\Phi^\circ \equiv \Phi^{(0)}_{a,b}$ and the remainder

$$\begin{aligned}
\Theta_{a,b,n_0}(t,\alpha,\beta) &:= t^{-n_0}(\Phi_{a,b} - \Phi_{a,b,n_0})(t,\alpha,\beta) \\
&= -\frac{\beta(z_a(t,\alpha) - z_b(t,\alpha-\beta))^{(n_0}}{(z_a(t,\alpha) - z_b(t,\alpha-\beta))(z_a(t,\alpha) - z_b(t,\alpha-\beta))^{n_0-1)}}.
\end{aligned}$$

Recall the expressions (2.14) and (2.15) that we obtained in chapter 2 for the Birkhoff-Rott operators $\mathscr{B}^\circ$ and $\mathscr{B}_a$ respectively. Then, we can express the Birkhoff-Rott operators $\mathscr{B}_a$ as follows. At time $t=0$ we have

$$\mathscr{B}^{\circ*} = \frac{\varpi^\circ}{\partial_\alpha z^\circ} H_{\Phi^\circ}(\partial_\alpha z^\circ) - H_{\Phi^\circ}(\varpi^\circ) - \frac{1}{2}\frac{\varpi^\circ}{\partial_\alpha z^\circ}, \tag{4.81}$$

and for $t>0$ we have

$$\mathscr{B}_a^* = \frac{1}{2}\sum_{b=\pm}\left(\frac{\varpi_b}{\partial_\alpha z_b} H_{\Phi_{a,b}}(\partial_\alpha z_b) - H_{\Phi_{a,b}}(\varpi_b) - \theta_{a,b}\frac{\varpi_b}{\partial_\alpha z_b}\right). \tag{4.82}$$

Notice that Lemma 4.4.1 implies

$$\|\Phi_{a,b}\|_{L^\infty} \leq \frac{4|\beta|}{\sqrt{\mathcal{C}(z^\circ)^{-2}|\beta|^2 + t^2 c(s)^2(b-a)^2}} \leq 4\mathcal{C}(z^\circ), \tag{4.83}$$

and also

$$\begin{aligned}
&|\Phi_{a,b}(t,\alpha,\beta)-\Phi_{a,b}(t,\alpha',\beta)| \qquad (4.84)\\
&=\left|\frac{\beta(\beta\Delta_\beta(z_b(t,\alpha)-z_b(t,\alpha'))+t(b-a)((c\tau^\perp)(\alpha)-(c\tau^\perp)(\alpha')))}{(z_a(t,\alpha)-z_b(\alpha-\beta,t))(z_a(\alpha',t)-z_\mu(\alpha'-\beta,t))}\right|\\
&\le 4^2\mathcal{C}(z^\circ)(\mathcal{C}(z^\circ)|\partial_\alpha z_b|_\delta+|b-a|c(\alpha)^{-1}|c\tau|_\delta)|\alpha-\alpha'|^\delta. \qquad (4.85)
\end{aligned}$$

Therefore, $\|\Phi_{a,b}\|_{0,\delta}$ is bounded. Similar estimates apply to $\|\Phi_{a,b,n_0}\|_{0,\delta}$ and $\|\Theta_{a,b,n_0}\|_{0,\delta}$. In light of Theorem 4.4.1 this proves:

Corollary 4.4.1. *For any fixed $0\le\delta'<\delta<1$ and any $a,b\in\{-1,1\}$, the operators $H_{\Phi_{a,b}}$, $H_{\Phi_{a,b,n_0}}$, $H_{\Theta_{a,b,n_0}}$ are bounded operators from $C^{0,\delta}$ to $C^{0,\delta'}$, with operator norm bounded in terms of $\mathcal{C}(z^\circ)$, $\|1/c\|_{L^\infty}$, the $C^{0,\delta}$-norm of ϖ and the $C^{1,\delta}$-norm of z and $c\tau^\perp$.*

Based on this corollary we can consider expansions in t as follows. By considering the Taylor decompositions $f=f^{n_0-1)}+t^{n_0}f^{(n_0}$ of $f=\partial_\alpha z_\mu,\varpi_\mu$, we split

$$\begin{aligned}
H_{\Phi_{a,b}}f&=H_{\Phi_{a,b,n_0}}f^{n_0-1)}+t^{n_0}(H_{\Phi_{a,b,n_0}}f^{(n_0}+H_{\Theta_{a,b,n_0}}f)\\
&=H_{\Phi_{a,b,n_0}}f^{n_0-1)}+O(t^{n_0}),
\end{aligned}$$

where the $O(t^{n_0})$ is understood in the $C^{0,\delta'}$-norm. Applying this to (4.82) we obtain for any $0\le n\le n_0-1$

$$\begin{aligned}
\mathscr{B}_a^{(n)*}=\frac{1}{2}\sum_{b=\pm}\Bigg(&\sum_{n_1+n_2+n_3=n}\binom{n}{\vec n}\left(\frac{\varpi_a}{\partial_\alpha z_a}\right)^{(n_3)}(H_{\Phi_{a,b,n_0}}(\partial_\alpha z_b^{(n_2)}))^{(n_1)}\\
&-\sum_{n_1+n_2=n}\binom{n}{\vec n}(H_{\Phi_{a,b,n_0}}(\varpi_\mu^{(n_2)}))^{(n_1)}-\theta_{a,b}\left(\frac{\varpi_b}{\partial_\alpha z_b}\right)^{(n)}\Bigg).
\end{aligned}$$

Therefore, $\mathscr{B}_a^{(n)}$ depends on

$$f^{(n_2)},\ (H_{\Phi_{a,b,n_0}}f^{(n_2)})^{(n_1)},\qquad \text{for } f=\partial_\alpha z,\partial_\alpha(c\tau^\perp),\varpi \text{ and } 0\le n_1+n_2\le n.$$

In particular, since $(H_{\Phi_{a,b}}f)^{(0)}=H_{\Phi^\circ}f$ for $f\in C^{0,\alpha}(\mathbb{T})$ and

$$\frac{1}{2}\sum_{b=\pm}\theta_{a,b}=\frac{1}{2}+\frac{a}{4},$$

we deduce the following corollary.

Corollary 4.4.2. *Assume $\partial_\alpha z,\varpi\in C^1(0,T;C^\alpha(\mathbb{T}))$ for some $\alpha>0$. Then,*

$$\mathscr{B}_a^{(0)}=\mathscr{B}^\circ-\tfrac{a}{4}\varpi^\circ\partial_\alpha z^\circ,$$

with $\mathscr{B}^\circ$ given in (2.14).

The operator $H_{\Phi_{a,b}}$

In this section we analyze $H_{\Phi_{a,b,n_0}}$. For ease of notation let us abbreviate from now on Φ_{a,b,n_0} by $\Phi_{a,b}$ and $z_a^{n_0-1)}$ by z_a. For any $0 \leq n_1 \leq n_0 - 1$, Faà di Bruno's formula yields

$$\partial_t^{n_1} \Phi_{a,b} = \sum_{r \in \pi_{n_1}} F_r \Phi_{a,b}^{|r|+1} \beta^{-|r|} \prod_{m=1}^{n_1} (\partial_t^m (z_a - z_b'))^{r_m}, \tag{4.86}$$

with $\pi_{n_1} := \{r \in \mathbb{N}_0^{n_1} \,:\, r_1 + 2r_2 + \cdots + n_1 r_{n_1} = n_1\}$, $|r| := r_1 + \cdots r_{n_1}$ and F_r the combinatorial constant

$$F_r := (-1)^{|r|} \frac{|r|! n_1!}{r_1! 1!^{r_1} \cdots r_{n_1}! n_1!^{r_{n_1}}}.$$

By writing $z_a - z_b' = (a-b)\nu + \delta_\beta z_b$ with $\nu \equiv itc\tau$, the binomial theorem yields

$$\begin{aligned}
&\prod_{m=1}^{n_1} (\partial_t^m (z_a - z_b'))^{r_m} \\
&= \prod_{m=1}^{n_1} \sum_{s_m=0}^{r_m} \binom{r_m}{s_m} ((a-b)\partial_t^m \nu)^{s_m} (\partial_t^m \delta_\beta z_\mu)^{r_m - s_m} \\
&= \sum_{s \leq r} (a-b)^{|s|} \beta^{|r-s|} \prod_{m=1}^{n_1} \binom{r_m}{s_m} (\partial_t^m \nu)^{s_m} (\partial_t^m \Delta_\beta z_b)^{r_m - s_m}.
\end{aligned}$$

Thus, by using (4.83), the dominated convergence theorem yields

$$\partial_t^{n_1} H_{\Phi_{a,b}} f = H_{\partial_t^{n_1} \Phi_{a,b}} f = \sum_{r \in \pi_{n_1}} F_r \sum_{s \leq r} V_{s,r} I_{s,r}, \quad t > 0, \tag{4.87}$$

where

$$V_{s,r} := (a-b)^{|s|} \prod_{m=1}^{n_1} \binom{r_m}{s_m} (\partial_t^m \nu)^{s_m}, \qquad I_{s,r} := \int_{\mathbb{T}} \Delta_\beta f \Phi_{a,b}^{|r|+1} \beta^{-|s|} \prod_{m=1}^{n_1} (\partial_t^m \Delta_\beta z_b)^{r_m - s_m} \, \mathrm{d}\beta.$$

Notice that, since $\nu = itc\tau$,

$$V_{s,r} = \begin{cases} (a-b)^{s_1} \dbinom{r_1}{s_1} (ic\tau)^{s_1}, & s = (s_1, 0), \\ 0, & \text{otherwise}. \end{cases} \tag{4.88}$$

Let us analyze the cases $a = b$ and $a \neq b$ separately. As we shall see, while for $a = b$ the expression (4.87) holds at $t = 0$ too, for $a \neq b$ this is only true for $n_1 = 0$.

Case $a = b$

Proposition 4.4.1. *For all $0 \leq \delta' < \delta$ and $f \in C^{k,\delta}(\mathbb{T})$ it holds that*

$$\|H_{\Phi_{a,b}} f\|_{C_t^{n_1} C_\alpha^{k,\delta'}} \leq \frac{C}{\delta - \delta'} \|f\|_{C^{k,\delta}}, \tag{4.89}$$

where C depends on $\mathcal{C}(z^\circ)$ and the $C^{k+1,\delta}$-norm of $z^{(n)}$, $\nu^{(n)}$ for $0 \le n \le n_0 - 1$. In particular,

$$\|(H_{\Phi_{a,b}} f)^{(n_1)}\|_{C^{k,\delta'}} \le \frac{C_0}{\delta - \delta'} \|f\|_{C^{k,\delta}}, \tag{4.90}$$

where C_0 depends on $\mathcal{C}(z^\circ)$ and the $C^{k+1,\delta}$-norm of $z^{(n)}$, $\nu^{(n)}$ for $0 \le n \le n_1$. Moreover,

$$(H_{\Phi_{a,a}} f)^{(n_1)} = H_{\Phi_{a,a}^{(n_1)}} f. \tag{4.91}$$

Proof. Since $a = b$ the expression (4.87) reads as

$$\partial_t^{n_1} H_{\Phi_{a,a}} f = \sum_{r \in \pi_{n_1}} F_r I_{0,r}. \quad t > 0. \tag{4.92}$$

Hence, for $k = 0$, (4.89) follows from Corollary 4.4.1. Moreover, (4.92) holds at $t = 0$ by the dominated convergence theorem, and thus (4.90)(4.91) follow from Corollary 4.4.1 too. For $k \ge 1$, the same holds by applying the Leibniz rule. □

Case $a \neq b$

Proposition 4.4.2. *For all $0 \le \delta' < \delta$ and $f \in C^{n_1+k,\delta}(\mathbb{T})$ we have*

$$\|H_{\Phi_{a,b}} f\|_{C_t^{n_1} C_\alpha^{0,\delta'}} \le \frac{C}{\delta - \delta'} \|f\|_{C^{n_1,\delta}}, \tag{4.93}$$

where C depends on $\mathcal{C}(z^\circ)$ and the $C^{1\vee(n_1-n),\delta}$-norm of $z^{(n)}$, $\nu^{(n)}$ for $0 \le n \le n_0 - 1$. In particular,

$$\|(H_{\Phi_{a,b}} f)^{(n_1)}\|_{C^{k,\delta'}} \le \frac{C_0}{\delta - \delta'} \|f\|_{C^{n_1+k,\delta}}, \tag{4.94}$$

where C_0 depends on $\mathcal{C}(z^\circ)$ and the $C^{k+1\vee(n_1-n),\delta}$-norm of $z^{(n)}$, $\nu^{(n)}$ for $0 \le n \le n_1$.

Proof. First of all recall Lemma 2.2.2.

Given $s \le r \in \pi_{n_1}$, let us show that $V_{s,r} I_{s,r}$ in (4.87) is bounded. If $s = 0$ we are done as in Proposition 4.4.1. Otherwise, similarly to Lemma 2.2.2, we split it as

$$V_{s,r} I_{s,r} = \sum_{j=1}^{|s|} \frac{\partial_\alpha^j f}{j!} J_{s,r,j} + J_{s,r},$$

where

$$\begin{aligned} J_{s,r,j} &:= V_{s,r} \int_{\mathbb{T}} \Phi_{a,b}^{|r|+1} \beta^{j-(|s|+1)} \prod_{m=1}^{n_1} (\partial_t^m \Delta_\beta z_b)^{r_m - s_m} \, \mathrm{d}\beta, \\ J_{s,r} &:= V_{s,r} \int_{\mathbb{T}} \Delta_\beta^{|s|+1} f \Phi_{a,b}^{|r|+1} \prod_{m=1}^{n_1} (\partial_t^m \Delta_\beta z_b)^{r_m - s_m} \, \mathrm{d}\beta. \end{aligned}$$

In view of (4.88), from now on we consider $s = (s_1, 0)$. In order to bound $J_{s,r}$ we need f to be in $C^{|s|,\delta}$ and notice that the largest $|s|$ is for $s_1 = r_1 = n_1$.

Fixed (s,r), the most singular term of $J_{s,r,j}$ is for $j=1$. Let us analyze it. If $s=r$ this is $J_{r,r,1}\propto C^{2,|r|+1}_{a,b}$ (recall Lemma 2.2.2). Otherwise, let m be the first index (indeed $m=1$) for which $s_m<r_m$. As we did for f, let us split $J_{s,r,1}$ as

$$\partial_t^m\Delta_\beta z_b=\sum_{n=m}^{n_0-1}\frac{t^{n-m}}{(n-m)!}\Delta_\beta z_b^{(n)}.$$

Thus, the terms with $n\geq|s|+m$ are bounded. For the terms with $n<|s|+m$ we split

$$\Delta_\beta z_b^{(n)}=\sum_{l=1}^{|s|+m-n}\frac{\partial_\alpha^l z_b^{(n)}}{l!}\beta^{l-1}+\beta^{|s|+m-n}\Delta_\beta^{|s|+m-n+1}z_b^{(n)}.$$

Then, since $r_m\geq1$ implies $|s|<|r|\leq n_1-(m-1)$, i.e. $|s|+m\leq n_1$, it is enough to impose that $z_b^{(n)}$ belongs to $C^{n_1-n,\delta}$. We repeat this $|r-s|$ times, where terms as in Lemma 2.2.2 appear and the worst is again $C^{2,|r|+1}_{a,b}$ with $|r|\leq n_1$. This concludes the proof for $k=0$. The case $k\geq1$ follows by applying the Leibniz rule at time $t=0$. □

4.5 Proof of the main results

As we mentioned in Remark 4.2.1, it can be seen that the velocity field (4.7), with p determined by the Bernoulli's law, is a weak solution to IE if and only if

$$b:=\partial_t\tilde{\varpi}\partial_\alpha z-\varpi(\partial_t z-\mathscr{B})=0, \tag{4.95}$$

or equivalently (4.9). Moreover, the jump in p along z vanishes in this case:

$$[p]=b\cdot\frac{\partial_\alpha z}{|\partial_\alpha z|^2}=0. \tag{4.96}$$

In our construction, the Reynolds stress R introduces a relaxation whereby (4.95) is regularized. As we saw in section 4.2, under our choice of $\bar{\omega}$ (4.28), the construction of R leading to a subsolution adapted to Ω_{tur} is subordinated to determine (z,ϖ) up to some order t^{n_0}. This was the key observation in [70] for the unstable Muskat problem. In light of Propositions 4.2.4 and 4.3.2, we have to take (z,ϖ) satisfying three pointwise conditions (c)-(e), which can be expressed compactly as

$$(\langle b\rangle-t\partial_\alpha(q\partial_\alpha z))^{2)}=0, \tag{4.97}$$

coupled with two average conditions (a)-b

$$\begin{aligned}\int\langle b\rangle^{(3}&=0,\\ \int(q_1|\partial_\alpha z|-c\{b\})^{(2}\cdot\partial_\alpha z^{\circ\perp}&=0,\end{aligned} \tag{4.98}$$

where

$$\langle b\rangle=\tfrac{1}{2}(\partial_t\tilde{\varpi}\partial_\alpha z-\varpi(\partial_t z-\langle\mathscr{B}\rangle)), \tag{4.99a}$$

$$\{b\}=\tfrac{1}{2}(\partial_t\tilde{\varpi}\partial_\alpha(c\tau^\perp)-\varpi(c\tau^\perp-\{\mathscr{B}\})), \tag{4.99b}$$

and $q = q_1 + iq_2$ satisfying

$$(q_1|\partial_\alpha z| - c\{b\})^{1)} \cdot \partial_\alpha z^{\circ\perp} = 0, \qquad q_2^{(0)} = c\{b\}^{(0)} \cdot \partial_\alpha z^\circ. \tag{4.100}$$

Let (z°, ϖ°) as in (4.3) for some $k_\circ \geq 0$ and $\delta > 0$ big enough. We define recursively $(z, \tilde{\varpi})$ by means of its Taylor decomposition

$$z(t,\alpha) := \sum_{n=0}^{n_0} \frac{t^n}{n!} z^{(n)}(\alpha) + t^{n_0+1} z^{(n_0+1}(t,\alpha),$$
$$\tilde{\varpi}(t,\alpha) := \sum_{n=1}^{n_0} \frac{t^n}{n!} \tilde{\varpi}^{(n)}(\alpha) + t^{n_0+1} \tilde{\varpi}^{(n_0+1}(t,\alpha),$$

starting from $(z, \tilde{\varpi})^{(0)} = (z^\circ, 0)$, namely the term of order $n = 0, 1, 2$ in (4.97) determines $(z, \tilde{\varpi})^{(n+1)}$, and so $n_0 = 3$ is enough.

Choice of $(z, \tilde{\varpi})^{(1)}$

The zero-order term of (4.97) reads as

$$\langle b \rangle^{(0)} = 0. \tag{4.101}$$

Since (4.99a) yields

$$\langle b \rangle^{(0)} = \tfrac{1}{2}(\tilde{\varpi}^{(1)} \partial_\alpha z^\circ - \varpi^\circ (z^{(1)} - \langle \mathscr{B} \rangle^{(0)})),$$

(4.101) is equivalent to

$$\varpi^\circ (z^{(1)} - \langle \mathscr{B} \rangle^{(0)}) \cdot \partial_\alpha z^{\circ\perp} = 0, \qquad \tilde{\varpi}^{(1)} = \varpi^\circ (z^{(1)} - \langle \mathscr{B} \rangle^{(0)}) \cdot \partial_\alpha z^\circ.$$

Thus, since $\langle \mathscr{B} \rangle^{(0)} = \mathscr{B}^\circ$, it is enough to set

$$z^{(1)} = \mathscr{B}^\circ, \qquad \tilde{\varpi}^{(1)} = 0. \tag{4.102}$$

In light of section 4.4, we have $z^{(1)} \in C^{k_\circ, \delta_1}(\mathbb{T}; \mathbb{R}^2)$ for any $0 < \delta_1 < \delta$.

Remark 4.5.1. Notice that (4.101) can be understood as that (4.95) must hold at $t = 0$. In particular (4.96) holds at $t = 0$ in the sense that there is no jump of p along z°

$$[p^{(0)}] = 2\langle b \rangle^{(0)} \cdot \partial_\alpha z^\circ = 0.$$

Choice of $(z, \tilde{\varpi})^{(2)}$

The first-order term of (4.97) reads as

$$\langle b \rangle^{(1)} = \partial_\alpha (q^{(0)} \partial_\alpha z^\circ). \tag{4.103}$$

On the one hand, (4.99a) and (4.102) yield

$$\langle b \rangle^{(1)} = \tfrac{1}{2}(\tilde{\varpi}^{(2)} \partial_\alpha z^\circ - \varpi^\circ (z^{(2)} - \langle \mathscr{B} \rangle^{(1)})).$$

On the other hand, since $|\partial_\alpha z^\circ| = 1$ implies $\partial_\alpha^2 z^\circ = \kappa^\circ \partial_\alpha z^{\circ\perp}$ with $\kappa^\circ := \partial_\alpha^2 z^\circ \cdot \partial_\alpha z^{\circ\perp} \equiv$ (signed) curvature of z°, we have

$$\partial_\alpha(q^{(0)}\partial_\alpha z^\circ) = (\partial_\alpha q^{(0)} + i\kappa^\circ q^{(0)})\partial_\alpha z^\circ.$$

In particular, since $\{\mathscr{B}\}^{(0)} = -\frac{1}{4}\varpi^\circ \partial_\alpha z^\circ$ and (4.99b) yield

$$\{b\}^{(0)} = -\tfrac{1}{2}\varpi^\circ(c\tau^\perp - \{\mathscr{B}\}^{(0)}) = -\tfrac{1}{2}\varpi^\circ(\tfrac{1}{4}\varpi^\circ + ic)\partial_\alpha z^\circ,$$

(5.76) implies

$$q^{(0)} = ic\{b\}^{(0)*}\partial_\alpha z^\circ = -\tfrac{1}{2}c\varpi^\circ(c + \tfrac{1}{4}i\varpi^\circ).$$

Therefore, (4.103) is equivalent to

$$\begin{aligned}\varpi^\circ(z^{(2)} - \langle\mathscr{B}\rangle^{(1)}) \cdot \partial_\alpha z^{\circ\perp} &= \tfrac{1}{4}\partial_\alpha(c(\varpi^\circ)^2) + \kappa^\circ c^2 \varpi^\circ,\\ \tilde{\varpi}^{(2)} - \varpi^\circ(z^{(2)} - \langle\mathscr{B}\rangle^{(1)}) \cdot \partial_\alpha z^\circ &= \tfrac{1}{4}\kappa^\circ c(\varpi^\circ)^2 - \partial_\alpha(c^2\varpi^\circ).\end{aligned}$$

Thus, it is enough to set

$$\begin{aligned}z^{(2)} &= \langle\mathscr{B}\rangle^{(1)} + (\tfrac{1}{2}c\partial_\alpha\varpi^\circ + \tfrac{1}{4}\varpi^\circ\partial_\alpha c + \kappa^\circ c^2)\partial_\alpha z^{\circ\perp},\\ \tilde{\varpi}^{(2)} &= \tfrac{1}{4}c\kappa^\circ(\varpi^\circ)^2 - \partial_\alpha(c^2\varpi^\circ).\end{aligned}$$

On the one hand, $\tilde{\varpi}^{(2)} \in C^{k_\circ - 1,\delta}(\mathbb{T};\mathbb{R})$. On the other hand, in light of section 4.4, we have $z^{(2)} \in C^{k_\circ - 1,\delta_2}(\mathbb{T};\mathbb{R}^2)$ for any $0 < \delta_2 < \delta_1$.

Remark 4.5.2. For $|\varpi^\circ| \gg 0$ one may set also $\tilde{\varpi}^{(2)} = 0$ by taking

$$z^{(2)} = \langle\mathscr{B}\rangle^{(1)} - \frac{2}{\varpi^\circ}\partial_\alpha(q^{(0)}\partial_\alpha z^\circ). \tag{4.104}$$

Furthermore, we may set $\tilde{\varpi} = 0$. For the Birkhoff-Rott equations (4.9) this can be done by taking $r = 0$, which can be understood as fixing the parametrization that keeps ϖ constant in time. Moreover, in this case one may assume also that ϖ° is constant in α by choosing z° properly (not necessarily arc-length). However, for mixed sign vorticities the choice (4.104) is singular. In spite of this, since $(\partial_\alpha(q^{(0)}\partial_\alpha z^\circ)) \cdot \partial_\alpha z^{\circ\perp} \sim \varpi^\circ$, it is not necessary to divide by ϖ° by taking $\tilde{\varpi}^{(2)}$ as above.

Choice of $(z, \tilde{\varpi})^{(3)}$

The second-order term of (4.97) reads as

$$\langle b\rangle^{(2)} = \partial_\alpha(q\partial_\alpha z)^{(1)}. \tag{4.105}$$

On the one hand, (4.99a) and (4.102) yield

$$\langle b\rangle^{(2)} = \tfrac{1}{2}(\tfrac{1}{2}\tilde{\varpi}^{(3)}\partial_\alpha z^\circ + \tilde{\varpi}^{(2)}\partial_\alpha z^{(1)} - \tfrac{1}{2}\varpi^\circ(z^{(3)} - \langle\mathscr{B}\rangle^{(2)})).$$

On the other hand, we split

$$\partial_\alpha(q\partial_\alpha z)^{(1)} = \underbrace{\partial_\alpha(q^{(0)}\partial_\alpha z^{(1)} + q_1^{(1)}\partial_\alpha z^\circ)}_{\equiv\tilde{q}} + \partial_\alpha(iq_2^{(1)}\partial_\alpha z^\circ) = \tilde{q} + (i\partial_\alpha q_2^{(1)} - \kappa^\circ q_2^{(1)})\partial_\alpha z^\circ.$$

In particular, since (4.99b) and (4.102) yield

$$\{b\}^{(1)} = \tfrac{1}{2}\varpi^\circ\{\mathscr{B}\}^{(1)},$$

(5.76) implies

$$q_1^{(1)} = -(q_1^{(0)}\partial_\alpha z^{(1)} + ic\{b\}^{(1)})\cdot\partial_\alpha z^\circ = \tfrac{1}{2}c\varpi^\circ(c\partial_\alpha z - i\{\mathscr{B}\})^{(1)}\cdot\partial_\alpha z^\circ.$$

Since $q_2^{(1)}$ is free, (4.105) is equivalent to solve

$$\int(\langle b\rangle^{(2)} - \tilde{q})\cdot\partial_\alpha z^{\circ\perp} = 0.$$
$$(\langle b\rangle^{(2)} - \tilde{q})\cdot\partial_\alpha z^\circ = -\kappa^\circ\underbrace{\int_0^\alpha(\langle b\rangle^{(2)} - \tilde{q})\cdot\partial_\alpha z^{\circ\perp}\,\mathrm{d}\alpha_1}_{=q_2^{(1)}}.$$

Thus, it is enough to set

$$\begin{aligned} z^{(3)} &= \langle\mathscr{B}\rangle^{(2)} + 2H\partial_\alpha z^{\circ\perp}, \\ \tilde{\varpi}^{(3)} &= -2h\cdot\partial_\alpha z^\circ - 2\kappa^\circ\int_0^\alpha(h\cdot\partial_\alpha z^{\circ\perp} - H\varpi^\circ)\,\mathrm{d}\alpha_1, \end{aligned}$$

with

$$h \equiv \tilde{\varpi}^{(2)}\partial_\alpha z^{(1)} - \tilde{q}, \quad H \equiv \frac{\int h\cdot\partial_\alpha z^{\circ\perp}}{\int(\varpi^\circ)^2}\varpi^\circ.$$

In light of section 4.4, it follows that $\tilde{\varpi}^{(3)} \in C^{k_\circ-2,\delta_3}(\mathbb{T};\mathbb{R})$ and $z^{(3)} \in C^{k_\circ-2,\delta_3}(\mathbb{T};\mathbb{R}^2)$ for any $0 < \delta_3 < \delta_2$. Therefore, Lemma 4.3.3 requires $k_\circ \geq 4$.

Choice of the remainder

The average conditions (4.98) read as

$$\begin{aligned} \int\langle b\rangle^{(3} &= 0, \\ \int(\langle b\rangle\cdot z + ct\{b\}\cdot\partial_\alpha z^{\circ\perp})^{(3} &= 0. \end{aligned} \tag{4.106}$$

We declare $\tilde{\varpi}^{(n_0+1} = 0$ and

$$z^{(n_0+1}(t,\alpha) := t^{-(n_0+1)}\int_0^t s^{n_0}S(s,\alpha)\,\mathrm{d}s = \int_0^1 s^{n_0}S(st,\alpha)\,\mathrm{d}s,$$

for some S to be determined. Since $\varpi^\circ \neq 0$ we can take a cutoff function $\psi^\circ \in C^\infty(\mathbb{T};\mathbb{R}_+)$ vanishing on $\{\alpha\in\mathbb{T} : |\varpi^\circ(\alpha)| \leq \frac{1}{2}\|\varpi^\circ\|_{L^\infty}\}$ and with $\int\psi^\circ = 1$. We declare

$$S := \frac{1}{\varpi^\circ}(\gamma\psi^\circ - \sigma\partial_\alpha(\psi^\circ\partial_\alpha z^\circ)), \tag{4.107}$$

with $(\gamma,\sigma)\in\mathbb{R}^2\times\mathbb{R}=\mathbb{R}^3$ a time dependent vector, to be determined. Let us split

$$\langle b\rangle^{(3}=\tfrac{1}{2}(\varGamma-\varpi^\circ S),$$
$$(\langle b\rangle\cdot z+ct\{b\}\cdot(\partial_\alpha z^{\circ\perp})^{(3}=\tfrac{1}{2}((\varGamma-\varpi^\circ S)\cdot z+\varLambda),$$

where

$$\varGamma\equiv\partial_\alpha\tilde{\varpi}S+(\partial_t\tilde{\varpi}\partial_\alpha z-\varpi(\partial_t z^{3)}-\langle\mathscr{B}\rangle))^{(3},$$
$$\varLambda\equiv 2(\langle b\rangle^{2)}\cdot z+ct\{b\}\cdot(\partial_\alpha z^{\circ\perp})^{(3}.$$

Therefore, (4.106) reads as

$$\gamma(t)=\int\varGamma(t),\qquad \sigma(t)=\int\varSigma(t), \tag{4.108}$$

with

$$\varSigma\equiv\frac{\varLambda+(\varGamma-\gamma\psi^\circ)\cdot z}{\displaystyle\int\psi^\circ\partial_\alpha z^\circ\cdot\partial_\alpha z}.$$

Hence, since (4.108) is an implicit equation $(\gamma,\sigma)=F(\gamma,\sigma)$ with $F\in C^{0,1}(\mathbb{R}^3;\mathbb{R}^3)$ and

$$|F(\gamma_1,\sigma_1)-F(\gamma_0,\sigma_0)|\leq O(t)|(\gamma_1,\sigma_1)-(\gamma_0,\sigma_0)|,$$

the existence (and uniqueness) of (γ,σ) (and so $S\in C_tC_\alpha^{k_\circ-1}$) follows from the Banach fixed point theorem, namely the Lipschitz constant, and so the time of existence $T_1>0$, depends on $\|z^\circ\|_{C^{k_\circ+1,\alpha}}$, $\|\varpi^\circ\|_{C^{k_\circ,\alpha}}$, $\|1/c\|_{L^\infty}$ and $\mathcal{C}(z^\circ)$.

Once we have fixed $(z,\tilde{\varpi})$ and c (6.60), we define the turbulence zone $\Omega_{\mathrm{tur}}(t)$ and the vorticity $\bar{\omega}(t)$ by means of the map $z(t)$ (4.12) and (4.28) respectively, for all $0<t\leq T$ smaller than T_0 in Lemma 4.4.1 and T_1 above. Secondly, we define the velocity $\bar{v}=\mathscr{B}(\bar{\omega})$, the pressure $\bar{p}$ by means of the Bernoulli's law (4.39), and the Reynolds stress R as in Proposition 4.2.1. Then, Proposition 4.2.4 guarantees that R is uniformly bounded and so Theorem 6.2.1 applies. Finally, for the associated weak solutions, Proposition 4.3.3 yields Theorems 4.1.1-4.1.3.

4.6 Piecewise harmonic subsolutions

Following [70, sec. 5], we generalize the previous construction from sections 4.2-4.5 to the case of subsolutions $\bar{v}$ whose vorticity $\bar{\omega}$ is concentrated on $2N$ curves for $N\geq 1$.

Geometric setup

In light of Example 4.1.1, it seems suitable to consider the grid $\Lambda:=\{\lambda_j\,:\,1\leq|j|\leq N\}$ of $[-1,1]$ given by

$$\lambda_j=\mathrm{sgn}j\,\tfrac{2|j|-1}{|\Lambda|-1}.$$

Observe that $0<\lambda_1<\cdots<\lambda_N=1$ and $\lambda_{-j}=-\lambda_j$ for $1\leq j\leq N$. Given $\lambda\in\Lambda_+:=\{\lambda_j\,:\,1\leq j\leq N\}$, at each $t>0$ we define $\Omega^\lambda_{\mathrm{tur}}(t)$ as the annular region in $\mathbb{R}^2$ whose boundary is

$$\partial\Omega^\lambda_{\mathrm{tur}}(t)=z_{-\lambda}(t)\cup z_\lambda(t),$$

with $z_{\pm\lambda}(t) := z_{\pm\lambda}(t,\mathbb{T})$ parametrized by the map

$$z_{\pm\lambda}(t,\alpha) := z_\lambda(t,\alpha) \pm \lambda t c(\alpha)\partial_\alpha z^\circ(\alpha)^\perp,$$

where z_λ is an evolution of z° to be determined. We note that, following Lemma 4.4.1, there is no intersection of different $z_\lambda(t)$ for short times because we shall take $z_\lambda^{(1)} = \mathscr{B}^\circ$ independently of λ (cf. sec. 4.5). The turbulence zone is then $\Omega_{\mathrm{tur}} := \Omega_{\mathrm{tur}}^{\lambda_N} \supset \cdots \supset \Omega_{\mathrm{tur}}^{\lambda_1}$.

The velocity

The *ansatz* (4.28) is generalized here by

$$\bar{\omega}(t) := \frac{1}{|\Lambda|}\sum_{\lambda\in\Lambda} z_\lambda(t)^\sharp(\varpi_\lambda(t)\,\mathrm{d}\alpha),$$

with $\varpi_\lambda = \varpi^\circ + \partial_\alpha\tilde{\varpi}_\lambda$ and $(z,\tilde{\varpi})_{-\lambda} = (z,\tilde{\varpi})_\lambda$ for every $\lambda\in\Lambda_+$. More precisely, we define $(z,\tilde{\varpi})_\lambda$ by means of its Taylor decomposition

$$z_\lambda(t,\alpha) := \sum_{k=0}^{n_0}\frac{t^n}{n!}z_\lambda^{(n)}(\alpha) + t^{n_0+1}z_\lambda^{(n_0+1)}(t,\alpha), \qquad \tilde{\varpi}_\lambda(t,\alpha) := \sum_{n=0}^{n_0}\frac{t^n}{n!}\tilde{\varpi}_\lambda^{(n)}(\alpha),$$

with

$$z_\lambda^{(n_0+1)}(t,\alpha) := t^{-(n_0+1)}\int_0^t s^{n_0}S_\lambda(s,\alpha)\,\mathrm{d}s = \int_0^1 s^{n_0}S_\lambda(st,\alpha)\,\mathrm{d}s,$$

and S_λ to be determined.

Thus, for $t>0$ the velocity is given by $\bar{v}(t) := (K*\bar{\omega}(t))^*$,

$$\bar{v}(t,x)^* = \frac{1}{|\Lambda|}\sum_{\lambda\in\Lambda}\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{\varpi_\lambda(t,\alpha)}{x-z_\lambda(t,\alpha)}\,\mathrm{d}\alpha, \qquad x\neq z_\lambda(t,\alpha).$$

This $\bar{v}(t)$ is bounded, anti-holomorphic outside $\cup_\lambda z_\lambda(t)$ but with tangential discontinuities along $\cup_\lambda z_\lambda(t)$. Indeed, these limits $\bar{v}_\lambda^\pm(t,\alpha)$ are

$$\bar{v}_\lambda^\pm = \mathscr{B}_\lambda \mp \frac{1}{2|\Lambda|}\frac{\varpi_\lambda}{\partial_\alpha z_\lambda(t,\alpha)^*},$$

where $\mathscr{B}_\lambda \equiv \mathscr{B}_\lambda(z,\varpi)$ are the Birkhoff-Rott type operators

$$\mathscr{B}_\lambda(t,\alpha)^* = \frac{1}{|\Lambda|}\sum_{\mu\in\Lambda}\frac{1}{2\pi i}\mathrm{pv}\int_{\mathbb{T}}\frac{\varpi_\mu(t,\beta)}{z_\lambda(t,\alpha)-z_\mu(t,\beta)}\,\mathrm{d}\beta.$$

Notice that the pv is not necessary for $\mu\neq\lambda$ when $t>0$. Therefore,

$$[\![\bar{v}]\!]_\lambda = \mathscr{B}_\lambda, \qquad [\bar{v}]_\lambda = -\frac{1}{|\Lambda|}\frac{\varpi_\lambda}{\partial_\alpha z_\lambda^*}.$$

This $\mathscr{B}_\lambda$ can be written as

$$\mathscr{B}_\lambda^* = \frac{1}{|\Lambda|}\sum_{\mu\in\Lambda}\left(\frac{\varpi_\mu}{\partial_\alpha z_\mu}H_{\Phi_{\lambda,\mu}}(\partial_\alpha z_\mu) - H_{\Phi_{\lambda,\mu}}(\varpi_\mu) - \theta_{\lambda,\mu}\frac{\varpi_\mu}{\partial_\alpha z_\mu}\right),$$

with $\theta_{\lambda,\mu} := \frac{1+\mathrm{sgn}(\lambda-\mu)}{2}$. In particular, for $\lambda = \lambda_j$, it is straightforward to check that

$$\frac{1}{|\Lambda|}\sum_{\mu\in\Lambda}\theta_{\lambda,\mu} = \frac{1}{2} + \mathrm{sgn} j\frac{2|j|-1}{2|\Lambda|} = \frac{1}{2} + \lambda\bar{c}_N,$$

from which we generalize Corollary 4.4.2:

$$\mathscr{B}_\lambda^{(0)} = \mathscr{B}^\circ - \lambda\bar{c}_N\frac{\varpi^\circ}{\partial_\alpha z^\circ}. \tag{4.109}$$

Helmholtz decomposition of $\bar{v}$

Analogously to section 4.2, $\bar{v}$ can be written as

$$\bar{v} = \nabla\phi + \mathscr{C}K_{x_0}^*,$$

where

$$\mathscr{C}(t,x) = \frac{1}{|\Lambda|}\sum_{\lambda\in\Lambda}(1-\mathrm{Ind}_{z_\lambda(t)}(x))\int\varpi^\circ,$$

and ϕ is the (piecewise) harmonic function

$$\phi(t,x) = \frac{1}{|\Lambda|}\left(\sum_{\lambda\in\Lambda}\frac{1}{2\pi i}\int_{\mathbb{T}}\varpi_\lambda(t,\alpha)(\mathrm{L}_{z_\lambda(t,\alpha)}(x) - (1-\mathrm{Ind}_{z_\lambda(t)}(x))\mathrm{Log}(x-x_0))\,\mathrm{d}\alpha\right)_1 + O(t),$$

for $x \notin \cup_\lambda z_\lambda(t)$, where O can be chosen in such a way that

$$\partial_t\phi(t,x) = \frac{1}{|\Lambda|}\left(\sum_{\lambda\in\Lambda}\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{(\partial_\alpha z\partial_t\tilde{\varpi} - \partial_t z\varpi)_\lambda(t,\alpha)}{x-z_\lambda(t,\alpha)}\,\mathrm{d}\alpha\right)_1,$$

and so

$$[\partial_t\phi]_\lambda = \tfrac{1}{|\Lambda|}(\varpi\frac{\partial_t z\cdot\partial_\alpha z}{|\partial_\alpha z|^2} - \partial_t\tilde{\varpi})_\lambda.$$

The pressure

We define $\bar{p}$ outside $\cup_\lambda z_\lambda$ by means of the Bernoulli's law

$$\bar{p} := -\partial_t\phi - \tfrac{1}{2}|\bar{v}|^2 \quad \text{outside } \cup_\lambda z_\lambda.$$

Thus, analogously to Proposition 4.2.2, we have

$$[\bar{p}]_\lambda = \frac{1}{|\Lambda|}\left(\partial_t\tilde{\varpi} - \varpi(\partial_t z - \mathscr{B})_\lambda\cdot\frac{\partial_\alpha z_\lambda}{|\partial_\alpha z_\lambda|^2}\right).$$

The Reynolds stress

We define R as

$$R := \sum_{\lambda\in\Lambda_+} R^\lambda \mathbb{1}_{\Omega_{\mathrm{tur}}^\lambda}.$$

Analogously to Proposition 4.2.3, each R^λ must satisfy

$$\begin{aligned} \mathrm{div} R^\lambda &= 0 && \text{in } \Omega_{\mathrm{tur}}^\lambda, \\ \pm(R^\lambda \partial_\alpha z^\perp)_{\pm\lambda} &= i b_{\pm\lambda} && \text{on } z_{\pm\lambda}, \end{aligned}$$

where each b_λ is the boundary condition

$$b_\lambda = \tfrac{1}{|\Lambda|}(\partial_t \tilde{\varpi}\partial_\alpha z - \varpi(\partial_t z - \mathscr{B}))_\lambda.$$

On the one hand, by (4.109) and taking $(z,\tilde{\varpi})_\lambda^{(1)} = (\mathscr{B}^\circ, 0)$ as in section 4.5, we have

$$\langle b\rangle_\lambda^{(0)} = 0, \quad \{b\}_\lambda^{(0)} = -\tfrac{|\lambda|}{|\Lambda|}\varpi^\circ(\bar{c}_N\varpi^\circ + ic)\partial_\alpha z^\circ,$$

where $\langle b\rangle_\lambda := \frac{1}{2}(b_\lambda + b_{-\lambda})$ and $\{b\}_\lambda := \frac{1}{2}(b_\lambda - b_{-\lambda})$ for $\lambda \in \Lambda_+$.
On the other hand, analogously to Proposition 4.3.2, since

$$|\mathring{R}^{\lambda(0)}| = |\{b\}_\lambda^{(0)} - \tfrac{1}{2}\mathrm{tr} R^{\lambda(0)}\partial_\alpha z^\circ|,$$

by taking $H^\lambda = l_3^\lambda = 0$ and $q_2^{\lambda(0)} = c|\lambda|\{b\}_\lambda^{(0)}\cdot\partial_\alpha z^\circ$, this is minimized by $\mathrm{tr}R^{\lambda(0)} = 2\{b\}_\lambda^{(0)}\cdot\partial_\alpha z^\circ$:

$$|\mathring{R}^{\lambda(0)}| = |\{b\}_\lambda^{(0)}\cdot\partial_\alpha z^{\circ\perp}| = \tfrac{|\lambda|}{|\Lambda|}c|\varpi^\circ|.$$

Finally, since

$$(\mathring{R}\partial_\alpha z^\perp)_\lambda^{(0)} = i(\{b\}^{(0)} - \{b\}_\lambda^{(0)}\cdot\partial_\alpha z^\circ\partial_\alpha z^\circ) = -\{b\}_\lambda^{(0)}\cdot\partial_\alpha z^{\circ\perp}\partial_\alpha z^\circ = \tfrac{|\lambda|}{|\Lambda|}c\varpi^\circ\partial_\alpha z^\circ,$$

we obtain

$$\begin{aligned} &(\mathscr{B}\mathring{R}\partial_\alpha z^\perp + (|\mathring{R}| + e')\partial_t z\cdot\partial_\alpha z^\perp)_\lambda^{(0)} \\ &= (\mathscr{B}^\circ - \lambda\bar{c}_N\varpi^\circ\partial_\alpha z^\circ)\cdot(\tfrac{|\lambda|}{|\Lambda|}c\varpi^\circ\partial_\alpha z^\circ) + \tfrac{|\lambda|}{|\Lambda|}c|\varpi^\circ|(\mathscr{B}^\circ\cdot\partial_\alpha z^{\circ\perp} + \lambda c) \\ &= \tfrac{|\lambda|}{|\Lambda|}c|\varpi^\circ|(\lambda(c - \bar{c}_N|\varpi^\circ|) + B^\circ), \end{aligned}$$

where $B^\circ := \mathscr{B}^\circ\cdot((\mathrm{sgn}\varpi^\circ + i)\partial_\alpha z^\circ)$. The rest follows analogously to the case $N = 1$.

4.7 Infinite energy lemmas

In this section we prove two lemmas for (bounded) weak solutions which may not have finite kinetic energy $E(t) = \frac{1}{2}\|v(t)\|_{L^2}^2$.

Lemma 4.7.1. *Let D be the dissipation measure in Proposition 4.3.3. For any $M > 0$ let $\psi_M \in C_c^\infty(\mathbb{R}^2;\mathbb{R}_+)$ be a radial function with $\mathbb{1}_{B_M} \le \psi_M \le \mathbb{1}_{B_{M+1}}$ and $\|\nabla\psi_M\|_\infty \le 2$. Then,*

$$\langle D(t_2) - D(t_1), \mathbb{1}\rangle = \lim_{M\to\infty}\langle D(t_2) - D(t_1), \psi_M\rangle = \int_{\mathbb{R}^2}(e(t_1) - e(t_2))\,\mathrm{d}x.$$

Proof. Since ψ_M does not depend on t, Definition 4.1.2 reads as

$$\langle D(t_2)-D(t_1),\psi_M\rangle=\int_{t_1}^{t_2}\int_{\mathbb{R}^2}(e+p)\,v\cdot\nabla\psi_M\,\mathrm{d}x\,\mathrm{d}s-\int_{\mathbb{R}^2}(e(t_2)-e(t_1))\psi_M\,\mathrm{d}x. \tag{4.110}$$

Notice that

$$\int_{\mathbb{R}^2}||\bar v(t_2)|^2-|\bar v(t_1)|^2|\,\mathrm{d}x=\int_{\mathbb{R}^2}|\underbrace{(\bar v(t_2)+\bar v(t_1))}_{\sim(1+|x|)^{-1}}\cdot\underbrace{(\bar v(t_2)-\bar v(t_1))}_{\sim(1+|x|)^{-2}}|\,\mathrm{d}x<\infty,$$

where we have applied (2.2). Hence, the dominated convergence theorem allows to pass to the limit in the second term of (4.110). Finally, since $|\nabla\psi_M|\leq 2\mathbb{1}_{B_{M+1}\setminus B_M}$ and $v,(e+p)\sim(1+|x|)^{-1}$, we have

$$\int_{\mathbb{R}^2}|(e+p)v\cdot\nabla\psi_M|\,\mathrm{d}x\leq 2\int_{B_{M+1}\setminus B_M}|(e+p)v|\,\mathrm{d}x\lesssim\log\left(\frac{M+1}{M}\right)\to 0$$

as $M\to\infty$. □

Lemma 4.7.2 (Weak-strong uniqueness principle)**.** *Assume there is a strong solution* $(\mathbf{v},\mathbf{p})$ *to IE satisfying* $(\nabla_{\mathrm{sym}}\mathbf{v})^-\in L^1_tL^\infty$*. Then, if* (v,p) *is an admissible weak solution to IE with* $v^\circ=\mathbf{v}^\circ$ *and*

$$\int_0^t\int_{B_{M+1}\setminus B_M}|v-\mathbf{v}|^a|p-\mathbf{p}|^b\,\mathrm{d}x\,\mathrm{d}s\underset{M\to\infty}{\longrightarrow}0, \tag{4.111}$$

for $(a,b)=(2,0)$ *and* $(1,1)$*, necessarily* $(v,p)=(\mathbf{v},\mathbf{p})$*.*

Proof. Let $\psi\in C^1(\mathbb{R}^3;[0,1])$ be a test function with $\psi\equiv 1$ on $\operatorname{supp}D$. Let us consider the error

$$F_\psi(t):=\frac{1}{2}\int_{\mathbb{R}^2}|v(t)-\mathbf{v}(t)|^2\psi(t)\,\mathrm{d}x.$$

Hence, we deduce

$$\begin{aligned}
F_\psi&=\int_{\mathbb{R}^2}e\psi\,\mathrm{d}x+\int_{\mathbb{R}^2}\mathbf{e}\psi\,\mathrm{d}x-\int_{\mathbb{R}^2}v\cdot\mathbf{v}\psi\,\mathrm{d}x\\
&\leq\int_0^t\int_{\mathbb{R}^2}(e\partial_t\psi+(e+p)\,v\cdot\nabla\psi)\,\mathrm{d}x\,\mathrm{d}s+\int_{\mathbb{R}^2}e^\circ\psi^\circ\,\mathrm{d}x && (4.112)\\
&\quad+\int_0^t\int_{\mathbb{R}^2}(\mathbf{e}\partial_t\psi+(\mathbf{e}+\mathbf{p})\,\mathbf{v}\cdot\nabla\psi)\,\mathrm{d}x\,\mathrm{d}s+\int_{\mathbb{R}^2}\mathbf{e}^\circ\psi^\circ\,\mathrm{d}x && (4.113)\\
&\quad-\int_0^t\int_{\mathbb{R}^2}(v\cdot\partial_t(\mathbf{v}\psi)+v\otimes v:\nabla(\mathbf{v}\psi)+p\operatorname{div}(\mathbf{v}\psi))\,\mathrm{d}x\,\mathrm{d}s-\int_{\mathbb{R}^2}v^\circ\cdot\mathbf{v}^\circ\psi^\circ\,\mathrm{d}x, && (4.114)
\end{aligned}$$

where we have applied $\langle D,\psi\rangle\geq 0$ in (4.112), $\langle\mathbf{D},\psi\rangle=0$ in (4.113) and Definition 4.1.1 for (v,p) tested with $\mathbf{v}\psi$ in (4.114). Since $\frac{1}{2}v^\circ\cdot\mathbf{v}^\circ=\mathbf{e}^\circ=e^\circ$, the last terms in (4.112)-(4.114) cancel each other out. It can be checked that the above inequality can be written as $F_\psi\leq I_\psi+J_\psi$ where

$$\begin{aligned}
I_\psi&=-\int_0^t\int_{\mathbb{R}^2}(v\cdot\partial_t\mathbf{v}+v\otimes v:\nabla\mathbf{v})\psi\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\int_{\mathbb{R}^2}(\mathbf{p}v+\mathbf{e}(\mathbf{v}-v))\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}s,\\
J_\psi&=\frac{1}{2}\int_0^t\int_{\mathbb{R}^2}|v-\mathbf{v}|^2D_t\psi\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\int_{\mathbb{R}^2}(p-\mathbf{p})(v-\mathbf{v})\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}s,
\end{aligned}$$

being $D_t \equiv \partial_t + v\cdot\nabla$ the material derivative. For I_ψ, since $(\mathbf{v},\mathbf{p})$ is a strong solution we have

$$-\int_0^t\int_{\mathbb{R}^2} v\cdot\partial_t\mathbf{v}\psi\,\mathrm{d}x\,\mathrm{d}s = \int_0^t\int_{\mathbb{R}^2} v\cdot\mathrm{div}(\mathbf{v}\otimes\mathbf{v})\psi\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\mathbb{R}^2} v\cdot\nabla\mathbf{p}\psi\,\mathrm{d}x\,\mathrm{d}s.$$

Hence, by applying

$$v\cdot\mathrm{div}(\mathbf{v}\otimes\mathbf{v}) = \mathbf{v}\cdot\nabla\mathbf{v}v, \qquad v\otimes v:\nabla\mathbf{v} = v\cdot\nabla\mathbf{v}v,$$

we get

$$\int_0^t\int_{\mathbb{R}^2}(v\cdot\mathrm{div}(\mathbf{v}\otimes\mathbf{v}) - v\otimes v:\nabla\mathbf{v})\psi\,\mathrm{d}x\,\mathrm{d}s = \int_0^t\int_{\mathbb{R}^2}(\mathbf{v}-v)\cdot\nabla\mathbf{v}v\psi\,\mathrm{d}x\,\mathrm{d}s.$$

Thus, since $v,\mathbf{v}\in L^\infty_\sigma$ implies

$$\begin{aligned} 0 &= \int_{\mathbb{R}^2} v\cdot\nabla(\mathbf{p}\psi)\,\mathrm{d}x = \int_{\mathbb{R}^2} v\cdot\nabla\mathbf{p}\psi\,\mathrm{d}x + \int_{\mathbb{R}^2}\mathbf{p}v\cdot\nabla\psi\,\mathrm{d}x,\\ 0 &= \int_{\mathbb{R}^2}(\mathbf{v}-v)\cdot\nabla(\mathbf{e}\psi)\,\mathrm{d}x = \int_{\mathbb{R}^2}\mathbf{e}(\mathbf{v}-v)\cdot\nabla\psi\,\mathrm{d}x + \int_{\mathbb{R}^2}(\mathbf{v}-v)\cdot\nabla\mathbf{v}\mathbf{v}\psi\,\mathrm{d}x, \end{aligned}$$

we have

$$I_\psi = -\int_0^t\int_{\mathbb{R}^2}(\mathbf{v}-v)\cdot\nabla_{\mathrm{sym}}\mathbf{v}(\mathbf{v}-v)\psi\,\mathrm{d}x\,\mathrm{d}s.$$

In particular

$$I_\psi \leq 2\int_0^t \|(\nabla_{\mathrm{sym}}\mathbf{v})^-\|_{L^\infty}F_\psi\,\mathrm{d}s.$$

Now let us take ψ_M as in Lemma 4.7.1. Thus, by applying the decay hypothesis (4.111), for every $\varepsilon>0$ there is $M_0(\varepsilon)>0$ so that

$$J_{\psi_M}\leq\varepsilon \quad \text{for all } M\geq M_0.$$

Plugging all together, Grönwall's inequality yields

$$F_{\psi_M}(t)\leq\varepsilon\exp\left(2\int_0^t\|(\nabla_{\mathrm{sym}}\mathbf{v})^-\|_{L^\infty}\,\mathrm{d}s\right) \quad \text{for all } M\geq M_0.$$

Finally, the statement follows by taking $\limsup_{M\to\infty}$ above and making $\varepsilon\to 0$ after. □

4.8 Vortex-blob scheme

In this section we provide some numerical simulations with the aim of illustrating how these solutions may look like. To this end we consider the classical vortex-blob regularization ([19, 87]), which consists of desingularazing the Cauchy kernel $K(x)=\frac{1}{2\pi ix}$ by introducing a parameter $\varepsilon>0$ in the denominator

$$K_\varepsilon(x) := K(x)\frac{|x|^2}{|x|^2+\varepsilon^2} = \left(\frac{1}{2\pi}\frac{x^\perp}{|x|^2+\varepsilon^2}\right)^*.$$

As shown in [92], for vorticities in the Delort's class $\mathcal{M}^+(\mathbb{R}^2)\cap H^{-1}(\mathbb{R}^2)$ (and also for the mixed sign case by assuming certain control on the maximal vorticity function), this method yields weak solutions to the 2D incompressible Euler equations (as the parameters vanish properly).

Let $h \equiv$ time step and $S \equiv$ grid of $\mathbb{T}$. Here we take $\varepsilon = 0.002$, $h = 0.025$ and $|S| = 20000$ points. Thus, we consider the discrete ε-BR equation

$$\frac{z(t+h,\alpha)-z(t,\alpha)}{h} = \frac{\ell_\circ}{2\pi|S|}\sum_{\beta\in S}\frac{(z(t,\alpha)-z(t,\beta))^\perp}{|z(t,\alpha)-z(t,\beta)|^2+\varepsilon^2}\varpi(t,\beta), \quad \alpha\in S, \tag{4.115}$$

which yields a recurrence for $t = 0,\ h,\ 2h,\ 3h,\ \ldots$ starting from z°. For simplicity we shall focus on the circle $z^\circ(\alpha) = e^{i\alpha}$ ($\ell_\circ = 2\pi$), for different vortex sheet strengths $\varpi(t) = \varpi^\circ$. To simulate the Kelvin-Helmholtz instability we consider a tiny perturbation of z°

$$z_{0,\gamma} = z^\circ - \gamma\partial_\alpha z^{\circ\perp},$$

with $\gamma(\alpha) = \epsilon\sin(k\alpha)$. Here we take $\epsilon = 0.001$ (perturbation amplitude) and $k = 30$ (perturbation frequency). For times $t = 0,\ 1.25$ and 2.5 (from top to bottom) we plot the macroscopic vector field

$$\bar{v}(t,x) = \frac{\ell_\circ}{2\pi|S||\Lambda|}\sum_{\lambda\in\Lambda}\sum_{\beta\in S}\frac{(x-z_\lambda(t,\beta))^\perp}{|x-z_\lambda(t,\beta)|^2+\varepsilon^2}\varpi(t,\beta),$$

the Kelvin-Helmholtz curve $z_\gamma(t)$ (light blue) given by (4.115) starting from $z_{0,\gamma}$, and the boundary of the turbulence zone $z_\pm(t) = z(t) \pm ct\partial_\alpha z^{\circ\perp}$ (dark blue) with $c(\alpha) = \beta(|\varpi^\circ| * \eta_\epsilon)(\alpha)$ and, for simplicity, $z(t)$ given by (4.115) starting from z°, coupled with the points where ϖ° vanishes (red). Here we take $|\Lambda| = 10$ and $\epsilon = \ell_\circ/20$. In the pictures below $\beta = \frac{1}{8}$. However, for short times we have observed that $\beta = \frac{1}{4}$ may fit better as ε decreases. Although we would have liked to explore the scope of this viewpoint in more detail, we have thought appropriate to present this simple approach here and leave possible improvements for future works.

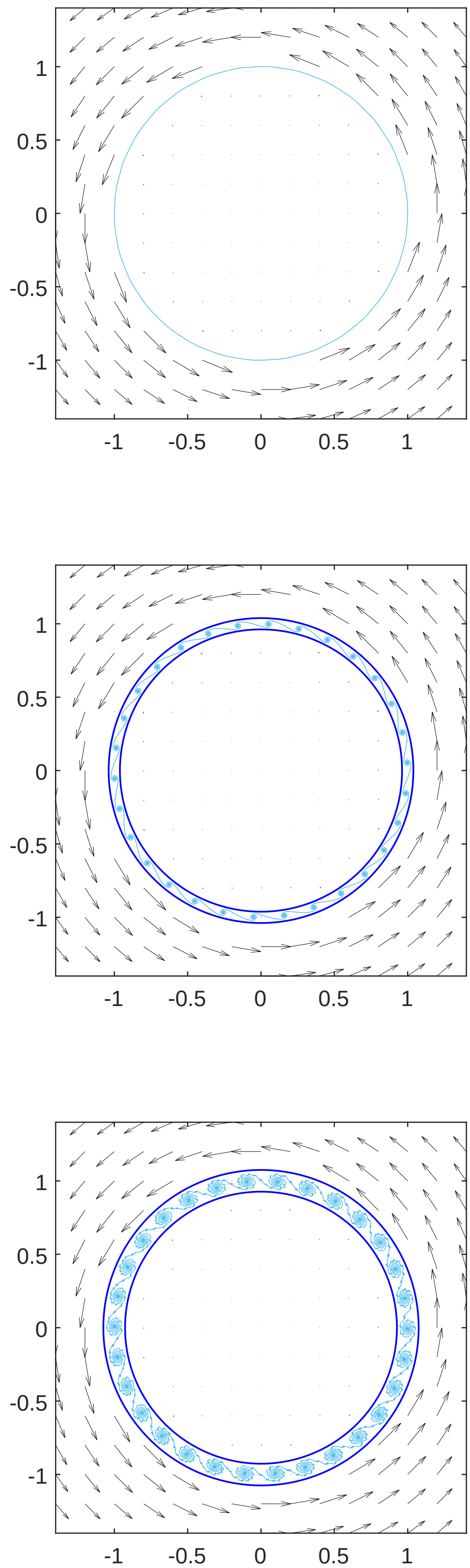


Figure 4.2: $\varpi^\circ(\alpha) = \frac{1}{4}$.

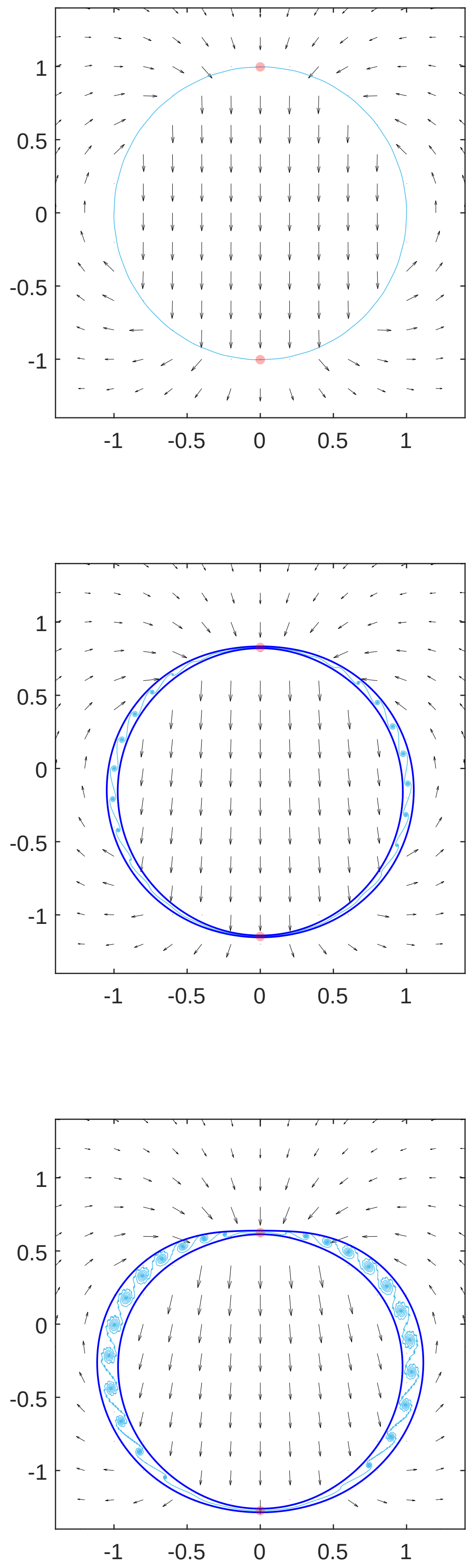


Figure 4.3: $\varpi^{\circ}(\alpha) = \frac{1}{4}\cos(\alpha)$.

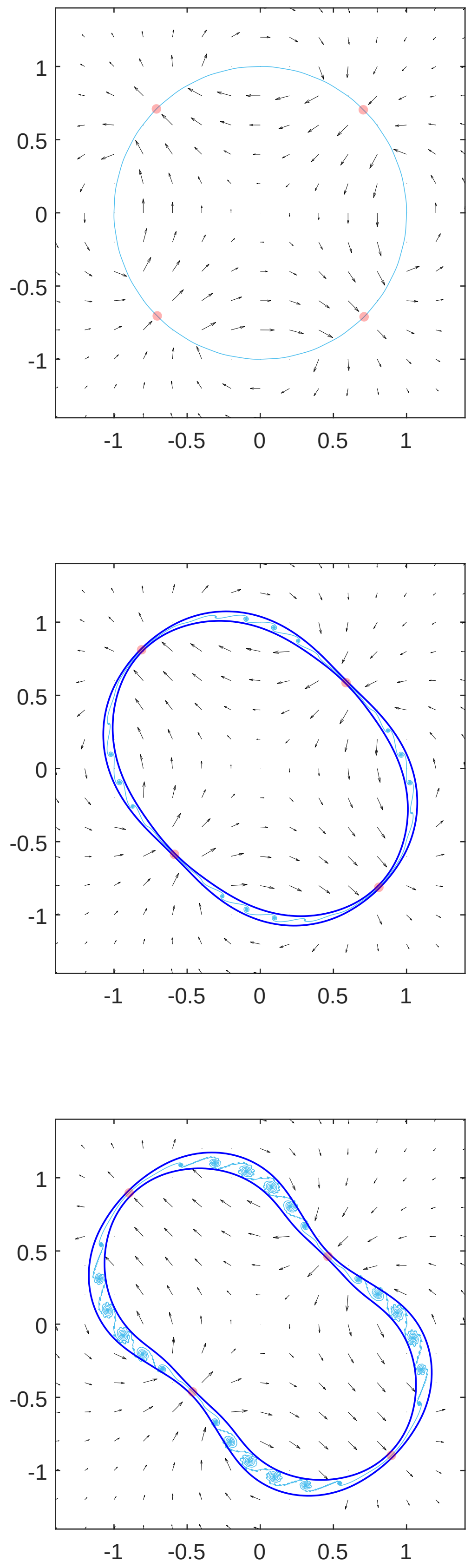


Figure 4.4: $\varpi^{\circ}(\alpha) = \frac{1}{4}\cos(2\alpha)$.

Chapter 5

H-principle for the IPM equation with density-viscosity jump

This chapter presents the paper [102].

5.1 Introduction and main results

We deal with the evolution of two incompressible fluids with constant densities[1] $\rho_h > \rho_l > 0$ and viscosities $\mu_h, \mu_l > 0$ (e.g. water and oil [110]) moving through a 2D porous medium $\mathscr{D}$ with constant permeability $\kappa > 0$ (or Hele-Shaw cell [119]) under the action of gravity[2] $-\rho g i$. Following [116], we introduce the $\{-1, 1\}$-valued variable $\theta(t, x)$ to indicate whether at time $t \in \mathbb{R}_+$ the pores near $x = (x_1, x_2) \in \mathscr{D}$ are filled with **phase** l or h:

$$a(t,x) := \frac{a_h + a_l}{2} + \frac{a_h - a_l}{2}\theta(t,x), \qquad a = \rho, \mu. \tag{5.1}$$

This two-phase flow can be modelled ([109]) by the incompressible porous media (IPM) equation

$$\partial_t \theta + \nabla \cdot (\theta v) = 0, \tag{5.2}$$

$$\mathrm{div} v = 0, \tag{5.3}$$

$$\tfrac{\mu}{\kappa} v = -\nabla p - \rho g i, \tag{5.4}$$

in $\mathbb{R}_+ \times \mathscr{D}$. (5.1)-(5.3) read as the phase distribution θ (resp. ρ and μ) is advected by the incompressible flow (coupled with the no-flux boundary condition). (5.4) is **Darcy's law**, which relates the velocity field v of the fluid with the forces acting on it. By renaming the pressure p, Darcy's law can be written in terms of the phase θ as

$$v + A_\mu \theta v + A_\rho \theta i = -\nabla p, \tag{5.5}$$

where A_ρ, A_μ are the **Atwood numbers**

$$A_\rho := \kappa g \frac{\rho_h - \rho_l}{\mu_h + \mu_l} > 0, \qquad A_\mu := \frac{\mu_h - \mu_l}{\mu_h + \mu_l} \in (-1, 1).$$

[1] $h \equiv$ heavier, $l \equiv$ lighter.

[2] $i \equiv (0, 1)$ by identifying $\mathbb{R}^2 \simeq \mathbb{C}$

Since (5.1)-(5.3) are invariant under the scaling $\theta(\alpha t, x)$, $\alpha v(\alpha t, x)$, by normalizing ($\alpha = A_\rho$) and renaming p, we may assume w.l.o.g. that $A_\rho = 1$. Thus, from now on we shall abbreviate $A \equiv A_\mu$. We shall refer to this system as IPM_A.

The main results

The phase jump induces Rayleigh-Taylor (RT) and vorticity at the interface separating both fluids, which becomes unstable when the RT condition fails (cf. sec. 5.1.1). In such a case, the two fluids can start to mix on a mesoscopic scale (see e.g. [134, pp. 261-267] and [80]).

In this chapter we investigate the scope of the convex integration viewpoint to the RT instability for IPM_A in the case of different viscosities (or mobilities in [116], cf. sec. 5.6) which is a recurrent theme in the applied literature. In short terms, the approach seems to work at least for flat interfaces, but the relaxation presents some unexpected singularities which makes the project challenging.

Before going any further let us present the problem discussed, summarize the main results of this chapter as well as the technical difficulties, and go back at the end of the introduction with a new link between the mixing regime and the relaxation. Firstly, we present two theorems regarding weak solutions to IPM_A for any $|A| < 1$ (cf. Def. 5.2.1). The first one exhibits lack of uniqueness in the class $C_t L^\infty_{w^*}$.

Theorem 5.1.1. *Let $|A| < 1$, $T > 0$ and $\mathscr{D} = \mathbb{R}^2$ or $\mathbb{T}^2$. There exist infinitely many weak solutions $(\theta, v) \in C(\mathbb{R}_+; L^\infty_{w*}(\mathscr{D}))$ to IPM_A with $|\theta| = 1$ on $(0,T) \times \mathscr{D}$ and $\theta = 0$ outside.*

Thus, IPM_A admits non-trivial weak solutions with compact support in time. Opposite to these paradoxical examples, we construct mixing solutions to the **unstable Muskat problem** with initial flat interface. This is IPM_A starting from the unstable planar phase

$$\theta^\circ(x) = \begin{cases} +1, & x_2 > 0, \\ -1, & x_2 < 0. \end{cases} \tag{5.6}$$

Similarly to [22, 26, 70, 126], we show that these weak solutions start to mix inside a mixing zone Ω_{mix} which grows linearly in time around $x_2 = 0$, and that they look macroscopically almost like the coarse-grained phase, denoted in this chapter by Θ_A (cf. (5.14)), introduced by Otto in [116]. For this reason, we shall call them "Θ_A-mixing solutions" (cf. Def. 5.2.3 and Fig. 5.6-5.11).

Theorem 5.1.2. *Let $|A| < 1$ and $\mathscr{D} = \mathbb{R}^2$ or $(-1,1)^2$. There exist infinitely many Θ_A-mixing solutions $(\theta, v) \in C(\mathbb{R}_+; L^\infty_{w*}(\mathscr{D}))$ to IPM_A starting from the unstable planar phase* (5.6).

While the weak solutions from Theorem 5.1.1 cannot attain the initial datum $\theta^\circ = 0$ in the strong sense, the ones from Theorem 5.1.2 satisfy $\theta \in C(\mathbb{R}_+; L^p_{\mathrm{loc}}(\mathscr{D}))$ for all $1 < p < \infty$. Moreover, they are forced to have finite mixing speed (cf. Prop. 5.2.1).

These theorems are deduced from a more general h-principle (cf. Thm. 5.2.1). Recalling chapter 3, this reads as weak solutions to IPM_A can be recovered via convex integration whenever a subsolution is provided (cf. sec. 5.2). This subsolution (cf. Def. 5.2.1) is a weak solution to a linearized version L_A of IPM_A, taking values in a relaxed set $\bar{U}_A$ of the corresponding constraint K, namely U_A is an open set satisfying a perturbation property w.r.t. (L_A, K).

The proof of the h-principle is classical ([26, 53, 126]) but difficulties arise as the parameter

A, which originally looks innocent, turns the relation between the components of the subsolution less explicit, which ends up hampering considerably the proof of the hypothesis (H1)-$(\mathrm{H3})_p$ required therein (cf. sec. 5.3). For instance, the L^p-boundedness property $(\mathrm{H3})_p$ becomes nontrivial for $0 < |A| < 1$ (cf. Lemmas 5.3.1 and 5.4.5). A more delicate issue is the relaxation $\bar{U}_A$. We take $\bar{U}_A = K^{lc,\Lambda_A} \equiv \Lambda_A$-lamination hull of K, which we compute explicitly (cf. (5.17) and sec. 5.4). However, since it is not obvious that such $\bar{U}_A$ is closed under weak*-convergence (not even that $\bar{U}_A$ is equal the functional Λ_A-convex hull of K) we refine the Baire category argument to adapt the proof of the h-principle we follow to our situation (cf. Rem. 5.3.1).

While the relaxation $\bar{U}_0$ only narrows at K, for different viscosities $\bar{U}_A$ develops a pinch singularity far away from K. Up to our knowledge, this kind of singularity outside the constraint K does not appear in other examples in Hydrodynamics. This necessarily complicates the existence of long Λ_A-segments as the perturbation property (H2) requires. To our surprise, they do exist even if $\bar{U}_A$ is very narrow far away from K. Remarkably, the use of Complex Analysis becomes very helpful, reducing considerably some tedious computations and providing a nice geometric interpretation in terms of the automorphisms of the unit disc (cf. Rem. 5.4.1).

In order to find bounded velocities, Székelyhidi computed cleverly the relaxation of some $K_M \Subset K$ for $A = 0$. In the case of viscosity jump the parameter A introduces an asymmetry that makes less clear what restriction of K may return a simple relaxation (cf. Rem. 5.4.2). The way of arguing is somewhat original as first we guess (inspired by an identity in [126]) a shape for $\bar{U}_{A,M}$, and then find $K_{A,M} \subset\subset K$ satisfying $(K_{A,M})^{lc,\Lambda_A} = \bar{U}_{A,M}$.

The proof of the perturbation property (H2) for $U_{A,M}$ presents some added difficulties compared to $A = 0$ (cf. Lemma 5.4.7). The main obstacle is that one of the inequalities bounding $U_{A,M}$, which is just a restriction on v for $A = 0$, depends on m (relaxation of the momentum θv) for $0 < |A| < 1$. Geometrically, the projection $U_{A,M}(\theta, v) \equiv \{m \in \mathbb{R}^2 : (\theta, v, m) \in U_{A,M}\}$, which is given by the intersection of three balls for $A = 0$, is also restricted by a half-plane for $0 < |A| < 1$ (cf. Fig. 5.1). This causes that $U_{A,M}(\theta, v)$ collapses as $|v|$ grows, in contrast to the case $A = 0$ (cf. Fig. 5.2-5.3). Furthermore, the pinch singularity becomes further complicated since the new inequalities defining $U_{A,M}$ can interfere with it (cf. Rem. 5.4.3). All this makes the choice of the Λ_A-segments cumbersome in some of the cases (see e.g. (5.83)(5.84)).

5.1.1 A link between the mixing regime and the relaxation

The aim of this section is to analyse the physical implications of the pinch singularity that arises at U_A. In a nutshell, it prevents the two fluids from mixing wherever there is neither Rayleigh-Taylor nor vorticity (equiv. ∇p and v are continuous) at the interface. Let us explain this in more detail.

The Muskat problem describes IPM_A under the assumption that there is a time-dependent moveable interface $z(t)$ separating $\mathscr{D}$ in two disjoint open sets $\Omega_h(t)$ and $\Omega_l(t)$. Let us denote $f^{\uparrow}$ ($f^{\downarrow}$) by the limit of $f(z + \varepsilon\partial_\alpha z^{\perp})$ as $\varepsilon \uparrow 0$ ($\varepsilon \downarrow 0$), and also $[f] := f^{\uparrow} - f^{\downarrow}$ by the jump of $f = \theta$, v, p along z.

The Biot-Savart system (5.3)(5.5) determines p and v in terms of z and $[\theta]$. On the one hand, the incompressibility condition (5.3) implies that $v = \nabla^{\perp}\psi$ for some stream function ψ, and so the vorticity $\omega := \mathrm{rot} v = \Delta\psi$. On the other hand, by applying ∇^* on Darcy's law (5.5), we deduce that both Δp and $\Delta\psi$ are Dirac measures supported on z, namely

$$\Delta(p + i\psi) = (\sigma + i\varpi)\delta_z,$$

for some scalar functions $\sigma \equiv$ Rayleigh-Taylor and $\varpi \equiv$ vorticity strength. Thus, both p and ψ (and so v) are recovered from σ and ϖ respectively by means of Potential Theory, namely they are harmonic outside z and have well-defined traces. Moreover, p and ψ are continuous ($[p] = [\psi] = 0$) but have discontinuous gradients along z

$$[\nabla(p + i\psi)] = -i\frac{\sigma + i\varpi}{\partial_\alpha z^*},$$

and so $[v] = i[\nabla\psi]$. Observe $\sigma = -[\nabla p] \cdot \partial_\alpha z^\perp$ and $\varpi = [v] \cdot \partial_\alpha z$. Thus, (the jump along z of) Darcy's law (5.5) reads as

$$\varpi + \sigma i = -[\theta](A\bar{v} + i)^* \partial_\alpha z, \tag{5.7}$$

where $\bar{v} := \frac{1}{2}(v^\uparrow + v^\downarrow)$ is the **mean velocity** along z. Observe that both σ and ϖ vanish if and only if $A\bar{v} + i = 0$. As we shall see, these are precisely the states where U_A pinches.

Finally, (5.2) turns out to be a free boundary problem, namely z is driven by the Birkhoff-Rott integrodifferential equations

$$\partial_t z = \bar{v}(z) + r\partial_\alpha z, \qquad z|_{t=0} = z^\circ, \tag{5.8}$$

where r represents the re-parametrization freedom, $\bar{v}(z) = \mathscr{B}(z, \varpi(z))$ with

$$\mathscr{B}(z, \varpi)(t, \alpha)^* = \frac{1}{2\pi i}\mathrm{pv}\int \frac{\varpi(t, \beta)}{z(t, \alpha) - z(t, \beta)}\, \mathrm{d}\beta,$$

and, by (5.7), $\varpi(z)$ is given by the (implicit) equation $\varpi(z) = -[\theta](A\mathscr{B}(z, \varpi(z)) + i) \cdot \partial_\alpha z$. Similarly, $\sigma(z) = [\theta](A\mathscr{B}(z, \varpi(z)) + i) \cdot \partial_\alpha z^\perp$.

In brief, this Cauchy problem (5.8) for z is well-posed provided the Rayleigh-Taylor (also called Saffman-Taylor [119]) condition for the Muskat problem, $\sigma > 0$, holds ([7, 6, 37, 74, 101, 100, 123]). The geometric meaning of $\sigma(z) > 0$ is not evident since the dependence on z is highly implicit. The situation is simpler for equal viscosities ($A = 0$) or flat interfaces ($\bar{v} = 0$) because $[\theta]\partial_\alpha z_1 > 0$ just requires the heavier fluid to remain below the lighter. The Muskat problem for $A = 0$ has been widely studied in the literature (see the survey [72] and the references therein).

When the RT condition fails the free boundary can turn into a growing strip, $\Omega_{\mathrm{mix}} \equiv$ mixing zone, where the phases start to mix on a mesoscopic scale. In the last years this kind of mixing solutions have been constructed by means of convex integration in the RT unstable regime ([22, 26, 70, 126]). They are driven by a two-scale dynamic: one dealing with the evolution of the pseudo-interface, which may describe the macroscopic fingering phenomenon, and other dealing with the laminar-turbulent transition region Ω_{mix} around the pseudo-interface.

In [22, 70] the authors discovered that mixing solutions also exist in the RT stable regime provided the velocity is discontinuous, i.e. when $\varpi \neq 0$. Inspired by chapter 4, we speculate it may describe a turbulence zone of spiral vortices, usually observed in the Kelvin-Helmholtz instability. We remark in passing that, since there are initial data z° for which both (5.8) is solvable and mixing solutions exist, a main unsolved question is to identify a selection criterion among them which leads to a unique physical solution.

In short, it seems that the mixing phenomenon may be triggered at least by two mechanisms: $\sigma < 0$ or $\varpi \neq 0$. By (5.7), one of these is awake at some point of the interface $z(\alpha)$ if

$$-[\theta](A\bar{v}(z(\alpha)) + i)^* \partial_\alpha z(\alpha) \in \mathcal{M},$$

where $\mathcal{M} := \mathbb{R}^2 \setminus \mathcal{L} \equiv$ **mixing regime** and $\mathcal{L} := \{\varpi + \sigma i \,:\, \sigma \geq 0 = \varpi\}$. Conversely, the open half-line $\mathcal{L}^\circ = \{\varpi + \sigma i \,:\, \sigma > 0 = \varpi\}$ classifies the points where the interface is RT stable and there is no vorticity. Remarkably, we have found that the relaxation U_A (for different viscosities) excludes $\partial\mathcal{L} = \{0\}$: a pinch a singularity arises at $A\bar{v} + i = 0$ (cf. (5.17)) representing the points $z(\alpha)$ where $\sigma = 0 = \varpi$. In other words, this relaxation approach prevents the two fluids from mixing wherever both ∇p and v are continuous.

Organization of the chapter. We start section 5.2 recalling briefly the background of the problem. After this, we present the h-principle from which Theorems 5.1.1-5.1.2 are deduced. The proof of this h-principle appears in section 5.3. In section 5.4 we compute $\bar{U}_A$, $\bar{U}_{A,M}$ and show some of their properties. With the aim of figuring out how these Θ_A-mixing solutions may look like, we introduce a toy random walk in Appendix 5.5 (Fig. 5.6-5.11). Finally, we recall in Appendix 5.6 some properties of Θ_A as well as the transition to the stable planar phase in the confined domain $\mathscr{D} = (-1,1)^2$.

5.2 H-principle for IPM

Following [39, 126], we introduce a new variable m to encode the momentum θv. Thus, if we denote $u = (\theta, v, m) \in [-1,1] \times \mathbb{R}^2 \times \mathbb{R}^2$, this two-phase flow can be interpreted in the spirit of Tartar as (cf. 3.1)

$$\operatorname{div}_{(t,x)}(L_A(u)) = 0, \tag{5.9}$$

$$u \quad K\text{-valued}, \tag{5.10}$$

in $\mathbb{R}_+ \times \mathscr{D}$, that is, a linear differential system (5.9) coupled with a non-linear pointwise constraint (5.10), where $L_A : \mathbb{R}^5 \to \mathbb{R}^{3\times 3}$ is the linear map

$$L_A(u) := \left[\begin{array}{c|cc} \theta & m_1 & m_2 \\ \hline 0 & v_1 & v_2 \\ 0 & v_2 + Am_2 + \theta & -v_1 - Am_1 \end{array}\right], \tag{5.11}$$

and K is the ((t,x)-independent) **constraint**

$$K := \{(\theta, v, m) \,:\, |\theta| = 1,\, m = \theta v\}. \tag{5.12}$$

Notice that (L_A, K) is more demanding than IPM_A because this does not require $|\theta| = 1$.

Brief overview of the case $A = 0$

In [39], Córdoba, Faraco and Gancedo discovered that the convex integration method developed in [54] for the incompressible Euler equation could be adapted to prove lack of uniqueness in $L^\infty(\mathbb{R}_+ \times \mathbb{T}^2)$ for IPM_0. In addition, they noticed that, in contrast to [54], K^{Λ_0} does not agree with K^{co}. To overcome this extra difficulty the authors resorted to the theory of laminates. Remarkably, this result was generalized for a class of active scalar equations by Shvydkoy in [122] (see [82] for improvements of the regularity).

Later in [126] Székelyhidi computed explicitly $K^{\Lambda_0} = \bar{U}_0$, with U_0 the open set of states $u = (\theta, v, m) \in [-1,1] \times \mathbb{R}^2 \times \mathbb{R}^2$ given by

$$U_0 := \{(\theta, v, m) \,:\, |2(m - \theta v) + (1 - \theta^2)i| < (1 - \theta^2)\}, \tag{5.13}$$

thus providing an h-principle for IPM_0 (see [85] for a generalization in a class of active scalar equations). Another advantage of this computation is that it allows to identify compatible boundary and initial conditions in order to obtain admissible solutions, opposite to those paradoxical examples with compact support in time. As a promising application in evolution of microstructures, Székelyhidi constructed weak solutions in $L^\infty(\mathbb{R}_+ \times (-1,1)^2)$ to the unstable Muskat problem with initial flat interface $z^\circ(\alpha) = (\alpha, 0)$. Remarkably, he observed that the subsolution $\bar{\theta}_\alpha$ (for any $0 < \alpha < 1$, being $c = 2\alpha$ the rate of expansion of the mixing zone) that naturally arises in this scenario is closely related to the relaxation introduced in [116] (see also [77, 115]). In this paper Otto dealt with the general case $|A| < 1$. Since this is the motivation of this chapter, we have thought appropriate to sketch briefly this approach in section 5.5.

In short, after introducing a Lagrangian relaxation of IPM_A, Otto obtained a unique (relaxed) solution (cf. sec. 5.5-5.6)

$$\Theta_A(t,x) = \begin{cases} +1, & x_2 > c_A^+ t, \\ \frac{x_2 + At}{t + Ax_2 + \sqrt{(1-A^2)t(t+Ax_2)}}, & -c_A^- t < x_2 < c_A^+ t, \\ -1, & -c_A^- t > x_2, \end{cases} \qquad \text{where} \quad c_A^{\pm} = \frac{2}{1 \mp A}, \tag{5.14}$$

which aims to capture the macroscopic properties of (exact) solutions to IPM_A, thus giving a prediction of the actual shape and evolution of the mixing profile. This Θ_A is indeed the (unique) entropy solution ([116, (3.72)]) of the conservation law (or Burgers type equation)

$$\partial_t \Theta = \partial_{x_2}\left(\frac{1-\Theta^2}{1-\Theta A}\right), \qquad \Theta|_{t=0} = \theta^\circ. \tag{5.15}$$

The link between the approaches of Székelyhidi and Otto for $A = 0$ is given by

$$\bar{\theta}_\alpha(t) = \Theta(\alpha t), \quad t \in \mathbb{R}_+, \tag{5.16}$$

(for any $0 < \alpha < 1$) where $\Theta \equiv \Theta_0$. The interpretation given in [126] of (5.16) is that, although weak solutions are clearly not unique due to the symmetry breakdown, the uniqueness result of Otto can be understood as selecting the subsolution with maximal mixing zone (cf. Prop. 5.2.1).

At this point we remark that a natural question that arises here is if (5.16) defines a subsolution in the general case $|A| < 1$. As we shall see in Theorem 5.2.2, this is the case.

Continuing the overview of the case $A = 0$, Castro, Córdoba and Faraco [22] applied this h-principle to construct weak solutions to the unstable Muskat problem for non-flat interfaces $z^\circ(\alpha) = (\alpha, f^\circ(\alpha))$ with $f^\circ \in H^5(\mathbb{R})$, by taking the subsolution as $\bar{\theta}_\alpha(t,x) = \Theta(\alpha t, x - f(t,x_1)i)$ with f a suitable evolution of f_0. Moreover, they showed that these solutions indeed mix inside the mixing zone, thus justifying the name "mixing solution". In [70] Förster and Székelyhidi obtained a similar result for $f_0 \in C_*^{3,\gamma}(\mathbb{R})$ with a simpler proof by taking piecewise constant subsolutions approaching the linear profile of Θ adapted to f_0.

Recently, the h-principle presented in [54] was adapted in [26] (chapter 3) to measure, in terms of weak*-continuous quantities, the proximity of the weak solutions coming from the convex integration scheme to the subsolution $\bar{u}$, thus selecting those which retain more information from $\bar{u}$, thereby emphasizing the fact that the subsolution aims to be the macroscopic solution (cf. Rem. 5.2.2). For this reason, the authors called them "degraded mixing solutions" (here Θ_0-mixing solutions).

Our extension to the case $|A| < 1$

With the aim of generalizing these results, we follow chapter 3 to prove an h-principle for the system IPM_A, which additionally provides weak solutions in the stronger class $C_t L^\infty_{w*}$. In order to prove it we need to check three hypothesis. The first one (H1) is the existence of localized plane waves of (L_A), which is checked similarly to [39, 126].
The second and more delicate part of this chapter is to compute a large enough set $\bar{U}_A$ satisfying the perturbation property (H2). This is the Λ_A-lamination hull of K, $K^{lc,\Lambda_A} = \bar{U}_A$ with U_A the open set of states $u = (\theta, v, m) \in [-1, 1] \times \mathbb{R}^2 \times \mathbb{R}^2$ given by

$$U_A := \{(\theta, v, m) \,:\, |2(1 - \theta A)(m - \theta v) + (1 - \theta^2)(Av + i)| < (1 - \theta^2)|Av + i|\}. \tag{5.17}$$

Observe that (5.17) generalizes (5.13). Notice that each slice $U_A(\theta, v)$ is an (open) disc of radius proportional to $(1 - \theta^2)|Av + i|$. Thus, while for $A = 0$ the relaxation U_0 only narrows as $|\theta| \uparrow 1$ (i.e. u tends to K), for $0 < |A| < 1$ a pinch singularity arises at $Av + i = 0$ far away from K. As we saw in section 5.1.1, these are the states for which both σ and ϖ vanish.
The last one $(\mathrm{H3})_\infty$ requires finding bounded subsets $U_{A,M}$ of U_A satisfying (H2), which is further laborious than the unbounded case.

Before embarking on this task we present the statement of our h-principle and we prove Theorems 5.1.1-5.1.2 as corollaries.

Definition 5.2.1. Let $L^\infty_S(\mathscr{D})$ be the closed linear subspace of $L^\infty_{w*}(\mathscr{D})$ consisting of functions $u = (\theta, v, m)$ satisfying the Biot-Savart system

$$\int_{\mathscr{D}} v \cdot \nabla\phi \,\mathrm{d}x = 0, \quad \forall \phi \in C^1_c(\bar{\mathscr{D}}), \tag{5.18}$$

$$\int_{\mathscr{D}} (v + Am + \theta i) \cdot \nabla^\perp \psi \,\mathrm{d}x = 0, \quad \forall \psi \in C^1_c(\mathscr{D}). \tag{5.19}$$

Notice that (5.18) includes the no-flux boundary condition.

Let $\theta^\circ \in L^\infty(\mathscr{D}; [-1, 1])$ and $T > 0$. We say that $\bar{u} = (\bar{\theta}, \bar{v}, \bar{m}) \in C([0, T]; L^\infty_S(\mathscr{D}; \bar{U}_A))$, where U_A was defined in (5.17), is a **subsolution** to IPM_A starting from θ° if, at each $t \in [0, T]$,

$$\int_0^t \int_{\mathscr{D}} (\bar{\theta}\partial_t \phi + \bar{m} \cdot \nabla\phi) \,\mathrm{d}x \,\mathrm{d}\tau = \int_{\mathscr{D}} \theta(t)\phi(t) \,\mathrm{d}x - \int_{\mathscr{D}} \theta^\circ \phi^\circ \,\mathrm{d}x, \quad \forall \phi \in C^1_c(\mathbb{R}_+ \times \bar{\mathscr{D}}). \tag{5.20}$$

In particular, a pair $(\theta, v) \in C([0, T]; L^\infty_{w*}(\mathscr{D}; [-1, 1] \times \mathbb{R}^2))$ is a **weak solution** to IPM_A if $u = (\theta, v, \theta v)$ is a subsolution to IPM_A.

Let $\bar{u}$ be a subsolution to IPM_A and $\emptyset \neq \Omega_{\mathrm{mix}} \subset [0, T] \times \mathscr{D}$ open. We say that $\bar{u}$ is **strict** w.r.t. Ω_{mix} if it is perturbable inside

$$\bar{u} \in C(\Omega_{\mathrm{mix}}; U_A), \tag{5.21}$$

and exact outside

$$\bar{m} = \bar{\theta}\bar{v} \quad \text{outside} \quad \Omega_{\mathrm{mix}}. \tag{5.22}$$

In particular, we say that $\bar{u}$ is **admissible** w.r.t. Ω_{mix} if it satisfies (5.21), (5.22) and

$$|\bar{\theta}| = 1 \quad \text{outside} \quad \Omega_{\mathrm{mix}}. \tag{5.23}$$

Definition 5.2.2. In the setting of Theorem 5.2.1 below we need to fix some arbitrary $\gamma \in [0,1)$ and $D \in C(\Omega_{\mathrm{mix}}; (0,1])$. With them we define the error function w.r.t. Ω_{mix}

$$E(t,R) := \frac{1 \wedge |R|^\gamma}{|R|} \sup_{x \in R} D(t,x).$$

Recall Remark 3.3.1.

Theorem 5.2.1 (H-principle for IPM_A). *Let $|A| < 1$, $T > 0$, $\emptyset \neq \Omega_{\mathrm{mix}} \subset (0,T] \times \mathscr{D}$ open, E as in Def. 5.2.2 and consider $P(u) := v \cdot (v + Am + \theta i)$. Suppose there is a strict subsolution $\bar{u} = (\bar{\theta}, \bar{v}, \bar{m})$ to IPM_A w.r.t. Ω_{mix}. Then, there exist infinitely many weak solutions (θ, v) to IPM_A satisfying that, at each $t \in [0,T]$:*

(a) They agree with $\bar{u}$ outside Ω_{mix}

$$(\theta, v)(t) = (\bar{\theta}, \bar{v})(t) \quad \textit{in} \quad \mathscr{D} \setminus \Omega_{\mathrm{mix}}(t).$$

(b) For every (bounded) open $\emptyset \neq \Omega \subset \Omega_{\mathrm{mix}}(t)$,

$$\int_\Omega (1 - \theta(t,x)^2)\,\mathrm{d}x = 0 < \int_\Omega (1 - \theta(t,x))\,\mathrm{d}x \int_\Omega (1 + \theta(t,x))\,\mathrm{d}x.$$

(c) For every bounded rectangle $\emptyset \neq R \subset \Omega_{\mathrm{mix}}(t)$,

$$\left| \fint_R [F(u) - F(\bar{u})](t,x)\,\mathrm{d}x \right| \leq E(t,R),$$

for $F = \mathrm{id}$ or P, where $u = (\theta, v, \theta v)$.

In addition, if $\bar{u}$ is admissible w.r.t. Ω_{mix}, then $\theta \in C([0,T]; L^p_{\mathrm{loc}}(\mathscr{D}))$ for all $1 < p < \infty$.

The above h-principle allows to prove the Theorems 5.1.1-5.1.2 by taking a suitable subsolution $\bar{u}$. The choice of $\bar{u}$ for Theorem 5.1.1 is related to [39, 126], but in order to guarantee the weak*-continuity of the non-linearity θv we have chosen a time dependent $\bar{m}$.

Proof of Theorem 5.1.1. By Theorem 5.2.1, we consider $\Omega_{\mathrm{mix}} = (0,T) \times \mathscr{D}$ and $\bar{u} = (0,0,\bar{m})$ with $\bar{m} \in C([0,T]; \mathbb{R}^2)$ satisfying $\bar{m}(0) = \bar{m}(T) = 0$ and $|2\bar{m}(t) + i| < 1$ for all $t \in (0,T)$. ☐

Before writing the proof of the Theorem 5.1.2 let us reformulate it with the new terminology.

Theorem 5.2.2. *Let $|A| < 1$, $\mathscr{D} = \mathbb{R}^2$ and $0 < \alpha < 1$. Then $\bar{u}_{A,\alpha}$ with*

$$\bar{\theta}_{A,\alpha}(t) = \Theta_A(\alpha t), \quad t \in \mathbb{R}_+, \tag{5.24}$$

$\bar{v}_{A,\alpha} = 0$ and $\bar{m}_{A,\alpha}$ given by (5.27), is an admissible subsolution to IPM_A w.r.t.

$$\Omega_{\mathrm{mix}} = \{(t,x) \in \mathbb{R}_+ \times \mathscr{D} \,:\, -\alpha c_A^- t < x_2 < \alpha c_A^+ t\}. \tag{5.25}$$

For $\mathscr{D} = (-1,1)^2$ the same holds except that (5.25) is only valid until $\Omega_{\mathrm{mix}}(t)$ meets either the lower or upper boundary of $(-1,1)^2$. After this, $\Omega_{\mathrm{mix}}(t)$ starts to reduce until it ends up collapsing and the stable planar phase is reached (cf. sec. 5.6.1).

Definition 5.2.3. We say that the weak solutions (θ, v) coming from the h-principle applied to $\bar{u}_{A,\alpha}$ are Θ_A-**mixing solutions** to IPM_A starting from the unstable planar phase (5.6). For $\mathscr{D} = \mathbb{R}^2$ let us denote

$$\Omega_\pm = \{(t,x) \in \mathbb{R}_+ \times \mathscr{D} \,:\, \pm x_2 > \alpha c_A^\pm t\}. \tag{5.26}$$

As in Thm. 5.2.2, for $\mathscr{D} = (-1,1)^2$ (5.26) changes once $\Omega_{\mathrm{mix}}(t)$ hits either $x_2 = -1$ or 1 (cf. sec. 5.6.1).
Thus, at each $t \in \mathbb{R}_+$, these Θ_A-mixing solutions satisfy:

(a) Non-mixing outside Ω_{mix}:
$$(\theta, v)(t) = (\pm 1, 0) \quad \text{in } \Omega_\pm(t).$$

(b) Mixing inside Ω_{mix}: For every (bounded) open $\emptyset \neq \Omega \subset \Omega_{\mathrm{mix}}(t)$,
$$\int_\Omega (1 - \theta(t,x)^2)\,\mathrm{d}x = 0 < \int_\Omega (1 - \theta(t,x))\,\mathrm{d}x \int_\Omega (1 + \theta(t,x))\,\mathrm{d}x.$$

(c) Θ_A-macroscopic behavior: For every bounded rectangle $\emptyset \neq R = S \times tL \subset \Omega_{\mathrm{mix}}(t)$,
$$\left| \fint_R \theta(t,x)\,\mathrm{d}x - \langle L \rangle_{A,\alpha} \right| \leq E(t,R) \quad \text{where} \quad \langle L \rangle_{A,\alpha} := \fint_L \Theta_A(\alpha, x_2)\,\mathrm{d}x_2.$$

(d) For $f(\theta, v) = v$, θv and $P(\theta, v, \theta v)$, and every bounded rectangle $\emptyset \neq R \subset \Omega_{\mathrm{mix}}(t)$,
$$\left| \fint_R f(\theta, v)(t,x)\,\mathrm{d}x \right| \leq E(t,R).$$

Remark 5.2.1. The properties (a)(b) justify the adjective "mixing" and (c) the tag "Θ_A" (cf. Rem. (5.6.1) and Prop. (5.6.1) for a explicit computation of $\langle L \rangle_{A,\alpha}$). The property (d) shows that $\bar{v}_{A,\alpha} = 0$ can be interpreted as the macroscopic velocity too, and also that the "power balance" P (cf. [26, (14)]), which is a quadratic quantity, is almost preserved.

Remark 5.2.2. In [126, Rem. 5], the interpretation that $\bar{\theta}_{0,\alpha}$ represents the coarse-grained phase follows from the fact that there is a sequence of exact solutions $\theta_k \overset{*}{\rightharpoonup} \bar{\theta}_{0,\alpha}$. Here, the property c closes the diagram (3.6) in the sense that it provides an explicit relaxation for each exact solution separately. Schematically, if we denote $X_{A,\alpha}$ by the space of these Θ_A-mixing solutions with mixing speed α, then we have

$$\begin{tikzcd} X_{A,\alpha} \arrow[r, bend left, "\text{average}"] & \bar{\theta}_{A,\alpha} \arrow[l, bend left, "\text{h-principle}"] \end{tikzcd}$$

where the upper arrow means that $\bar{\theta}_{A,\alpha}$ can be recovered from each $\theta \in X_{A,\alpha}$ by averaging it over horizontal lines as follows

$$\bar{\theta}_{A,\alpha}(t,x) = \lim_{M \to \infty} \fint_{R_M(x)} \theta(t,x')\,\mathrm{d}x', \qquad (t,x) \in \mathbb{R}^2 \times \mathbb{R}_+,$$

with $R_M(x) = x + (-M, M) \times (-M^{-\delta}, M^{-\delta})$ for some arbitrary $\delta \in (0,1)$.

Proof of Theorem 5.2.2. Consider $\bar{\theta} = \bar{\theta}(t, x_2)$, $\bar{v} = 0$ and $\bar{m}$ to be determined. The condition (5.21) reads as $\bar{u}$ maps continuously Ω_{mix} into (see (5.17))

$$|2(1 - \bar{\theta}A)\bar{m} + (1 - \bar{\theta}^2)i| < (1 - \theta^2).$$

This suggests to take, for some $0 < \alpha < 1$,

$$\bar{m} = -\alpha \frac{1 - \bar{\theta}^2}{1 - \bar{\theta}A} i. \tag{5.27}$$

On the one hand, (5.18)(5.19) is automatically satisfied. On the other hand, (5.20) reads as

$$\partial_t \bar{\theta} = \alpha \partial_{x_2} \left(\frac{1 - \bar{\theta}^2}{1 - \bar{\theta}A} \right). \tag{5.28}$$

The (unique) entropy solution of the above scalar conservation law is (5.24). Finally, it is clear that $\bar{u}$ is admissible w.r.t. Ω_{mix}. □

We conclude this section by extending Prop. 4.3 in [126] to the general case $|A| < 1$. Roughly speaking this reads as, among subsolutions $\bar{u}$ to IPM_A starting from (5.6) with planar symmetry, the borderline case $\alpha = 1$ in Thm. 5.2.2 maximizes the mixing zone. As suggested in [126], this may serve as a selection criterion. We remark in passing that, inspired by chapter 4, the intermediate case $\alpha = \frac{1}{2}$, which maximizes the energy dissipation rate for the Kelvin-Helmholtz instability, may contain relevant physical information and then should be explored in future works.

Let us assume that $\partial_{x_1}\bar{u} = 0$ and that both fluids are at rest ($\bar{v} = 0$) outside Ω_{mix}. Then, (5.18)(5.19) implies that

$$\bar{v} = -A\bar{m}_1. \tag{5.29}$$

Notice that $A = 0$ yields $\bar{v} = 0$. Indeed, in [126] $\bar{v} = 0$ follows from the slighter assumption $\partial_{x_1}\bar{\theta} = 0$. Although Proposition 5.2.1 below holds in the class $\bar{v} = 0$ too, we find more natural the condition (5.29) here.

As in [126], on the confined domain $(-1, 1)^2$ the no-flux boundary condition implies $\bar{v} = 0$. Therefore, Prop. 4.3 in [126] can be extended analogously for $\mathscr{D} = (-1, 1)^2$. However, if we remove the vertical walls, say $\mathscr{D} = \mathbb{T} \times (-1, 1)$, then (5.29) requires some extra computations. Let us see it. Notice that $A\bar{v} + i \neq 0$ because $\bar{v}_2 = 0$. Then, since $\bar{u}$ is $\bar{U}_A$-valued (see (5.17)), the following inequality holds (a.e.)

$$\left| \frac{2(1 - \bar{\theta}A)(\bar{m} - \bar{\theta}\bar{v})}{A\bar{v} + i} + (1 - \bar{\theta}^2) \right| \leq (1 - \bar{\theta}^2). \tag{5.30}$$

By taking the real part of (5.30) and applying (5.29), we get

$$-\frac{1 - \bar{\theta}^2}{1 - \bar{\theta}A} \leq \left(\frac{\bar{m} - \bar{\theta}\bar{v}}{A\bar{v} + i} \right)_1 = -\frac{((\bar{m} + \bar{\theta}A\bar{m}_1)(A^2\bar{m}_1 + i))_1}{1 + (A^2\bar{m}_1)^2} = \frac{\bar{m}_2 - (1 + \bar{\theta}A)(A\bar{m}_1)^2}{1 + (A^2\bar{m}_1)^2},$$

and so

$$\begin{aligned} \bar{m}_2 &\geq -\frac{1 - \bar{\theta}^2}{1 - \bar{\theta}A}(1 + (A^2\bar{m}_1)^2) + (1 + \bar{\theta}A)(A\bar{m}_1)^2 \\ &= -\frac{1 - \bar{\theta}^2}{1 - \bar{\theta}A} + (A\bar{m}_1)^2 \frac{1 - A^2}{1 - \bar{\theta}A} \geq -\frac{1 - \bar{\theta}^2}{1 - \bar{\theta}A}. \end{aligned} \tag{5.31}$$

The rest follows similarly to [126]. Let us denote $\Omega_\pm = \{(t,x) \in \mathbb{R}_+ \times \mathscr{D} \,:\, \pm x_2 > c_A^\pm t\}$. By approximation, $\phi_\pm(t,x) = \max\{\pm x_2 - c_A^\pm t, 0\}$ is a valid test function. Then, since

$$c_A^\pm |\Omega_\pm| = 1 = \pm \int_{\mathscr{D}} \theta^\circ \phi_\pm^\circ \,\mathrm{d}x,$$

by evaluating (5.20) with $\phi_\pm$ we obtain

$$\int_{\Omega_\pm} (c_A^\pm (1 \mp \bar{\theta}) + \bar{m}_2) \,\mathrm{d}x \,\mathrm{d}t = 0.$$

Finally, since (5.31) implies

$$c_A^\pm (1 \mp \bar{\theta}) + \bar{m}_2 \geq (1 \mp \bar{\theta}) \left(\frac{2}{1 \mp A} - \frac{1 \pm \bar{\theta}}{1 - \bar{\theta} A} \right) = \frac{(1 \mp \bar{\theta})^2 (1 \pm A)}{(1 - \bar{\theta} A)(1 \mp A)} \geq 0,$$

necessarily $\bar{\theta} = \pm 1$ in $\Omega_\pm$. In summary, at least for bounded and rectangular $\mathscr{D}$'s (cf. [42]), either with or without vertical boundaries, the following holds.

Proposition 5.2.1. *Let $\bar{u}$ be a subsolution to* IPM_A *starting from* (5.6) *w.r.t. some* Ω_{mix} *and satisfying* (5.29). *Then $\bar{\theta} = \pm 1$ in $\Omega_\pm$, i.e. $\Omega_{\mathrm{mix}} \subset \{(t,x) \in \mathbb{R}_+ \times \mathscr{D} \,:\, -c_A^- t < x_2 < c_A^+ t\}$.*

5.3 Proof of the h-principle

In this section we prove Theorem 5.2.1. To this end, we need to check the following three hypothesis (cf. chapter 3). We do so for $p = \infty$ and also for $p = 2$ on $\mathscr{D} = \mathbb{T}^2$. Although $L^\infty(\mathbb{T}^2) \subset L^2(\mathbb{T}^2)$, the direct proof (Prop. 5.3.1) for $p = 2$ shows that $\bar{U}_A$ is somehow sharp.

(H1) Localized plane waves. Let $0 \neq h \in C^1(\mathbb{T}; [-1,1])$ with $\int h = 0$. There is a cone $\Lambda \subset \mathbb{R}^5$ so that, for all $\underline{u} \in \Lambda$ and $\psi \in C_c^\infty(\mathbb{R}^3)$ there is $\xi \in \mathbb{R} \times \mathbb{S}^1$ for which there are smooth solutions to (5.20)-(5.19) of the form

$$u_k(t,x) = \underline{u} h(k\xi \cdot (t,x)) \psi(t,x) + O(k^{-1}),$$

with $k \in \mathbb{N}$ and O depending on $|\underline{u}|, |\xi|$ and $\{|D^\alpha \psi(t,x)| \,:\, 1 \leq |\alpha| \leq 2\}$.

(H2) Long Λ-segments. There is an open set $U \subset [-1,1] \times \mathbb{R}^2 \times \mathbb{R}^2$ and a function $\phi \in C(]0,1];]0,1])$ such that, for all $u \in U$ there is $\underline{u} \in \Lambda$ with $\underline{\theta} = 1$ for which

$$u + \lambda \underline{u} \in U, \quad |\lambda| \leq \phi(1 - \theta^2).$$

(H3)$_p$ $C_t L_S^p$-boundedness. The space $L_S^p(\mathscr{D}; \bar{U})$ is L^p-bounded.

Let us start checking (H1). Since

$$\det(L_A(\underline{u})) = -\underline{\theta v} \cdot (\underline{v} + A\underline{m} + \underline{\theta} i) = \tfrac{1}{4} \underline{\theta} (|A\underline{m} + \underline{\theta} i|^2 - |2\underline{v} + A\underline{m} + \underline{\theta} i|^2),$$

from the definition of the wave cone (3.4) it follows that

$$\Lambda_A = \Lambda_0 \cup \Lambda_1, \tag{5.32}$$

with $\Lambda_j \equiv \Lambda_{A,j}$ given by

$$\Lambda_0 := \{\underline{u} \in \mathbb{R}^5 \,:\, \underline{\theta} = 0,\, \underline{v} = -A\underline{m}\},$$
$$\Lambda_1 := \{\underline{u} \in \mathbb{R}^5 \,:\, \underline{\theta} \neq 0,\, \underline{v} = \underline{\omega}(A\underline{m} + \underline{\theta} i) \text{ for some } \underline{\omega} \in \mathbf{S}\},$$

where

$$\mathbf{S} := \{\underline{\omega} \in \mathbb{R}^2 \,:\, |2\underline{\omega} + 1| = 1\},$$

that is, $\mathbf{S}$ is the sphere centered at $-\frac{1}{2}$ with radius $\frac{1}{2}$. We shall also consider the interior of its convex hull $\mathbf{D} = (\mathbf{S}^{\mathrm{co}})^\circ = \{\omega \in \mathbb{R}^2 \,:\, |2\omega + 1| < 1\}$. Both can be expressed in terms of the unit sphere $\mathbb{S}$ and the unit disc $\mathbb{D}$ as $\mathbb{S} = T\mathbf{S}$ and $\mathbb{D} = T\mathbf{S}$ where $T : \bar{\mathbf{D}} \to \bar{\mathbb{D}}$ is the translation

$$T\omega := 2\omega + 1. \tag{5.33}$$

Lemma 5.3.1. (H1) *holds for* $\Lambda = \Lambda_A$.

Proof. Step 1. Construction of a potential: Let us suppose that $u = (\theta, v, m)$ is a smooth localized solution to (5.20)-(5.19). Then, by (5.18), $v = \nabla^\perp f$ for some smooth f. If we write m in its Hodge's decomposition, $m = \nabla^\perp\varphi + \nabla g$ for some smooth φ, g, then (5.20) and (5.20)-(5.19) read as

$$\partial_t\theta + \Delta g = 0,$$
$$\Delta(f + A\varphi) + \partial_{x_1}\theta = 0.$$

Notice that $\theta = \Delta\phi$ for some smooth ϕ. Hence, $g = -\partial_t\phi$ and $f = -(\partial_{x_1}\phi + A\varphi)$. In summary,

$$\theta = \Delta\phi, \qquad v = -\nabla^\perp(\partial_{x_1}\phi + A\varphi), \qquad m = \nabla^\perp\varphi - \partial_t\nabla\phi.$$

This suggests to consider the following potential

$$P(\phi, \varphi) := (\Delta\phi, -\nabla^\perp(\partial_{x_1}\phi + A\varphi), \nabla^\perp\varphi - \partial_t\nabla\phi).$$

Since $v + Am + \theta i = \nabla(\partial_{x_2} - A\partial_t)\phi$, it satisfies $\mathrm{div}L_A(P(\phi, \varphi)) = 0$ for all $\phi, \varphi \in C^3(\mathbb{R}^3)$.

Step 2. Construction of u_k: Let us take $H \in C^3(\mathbb{T})$ such that $H'' = h$.
Given $\underline{u} = (\underline{\theta}, \underline{v}, \underline{m}) \in \Lambda$ and $k \in \mathbb{N}$, we consider

$$\phi_k(t, x) = \tfrac{a}{k^2}H(k\xi \cdot (t, x)), \quad \varphi_k(t, x) = \tfrac{b}{k}H'(k\xi \cdot (t, x)),$$

with $\xi = (\xi_0, \zeta) \in \mathbb{R} \times \mathbb{S}^1$ and $a, b \in \mathbb{R}$ to be determined. This choice yields

$$P(\phi_k, \varphi_k)(t, x) = (a, -i(a\zeta_1 + bA)\zeta, (bi - a\xi_0)\zeta)h(k\xi \cdot x).$$

Then, to prove (H1) we need to find ξ, a, b satisfying

$$(a, -i(a\zeta_1 + bA)\zeta, (bi - a\xi_0)\zeta) = (\underline{\theta}, \underline{v}, \underline{m}). \tag{5.34}$$

The first column in (5.34) reads as $a = \underline{\theta}$. Firstly assume that $\underline{u} = \Lambda_0$, i.e. $a = 0$ and $\underline{v} = -A\underline{m}$. Hence, the second and third column in (5.34) are equivalent to $\underline{m} = b\zeta^\perp$. Thus, we take $b = |\underline{m}|$ and $\zeta \in \mathbb{S}^1$ such that $\underline{m} = b\zeta^\perp$. Secondly assume that $\underline{u} \in \Lambda_1$, i.e. $a \neq 0$ and there is $\underline{\omega} \in \mathbf{S}$ so that $\underline{v} = \underline{\omega}(A\underline{m} + ai)$. Hence, for the third column in (5.34), $\underline{m} = (bi - a\xi_0)\zeta$, necessarily $\xi_0 = -a^{-1}\underline{m} \cdot \zeta$

and $b = \underline{m}\cdot\zeta^\perp$. Now, the second column in (5.34) reads as $\underline{v} = -i(a\zeta_1 + bA)\zeta = -\zeta^\perp(A\underline{m}+ai)\cdot\zeta^\perp$. Since $\underline{v} = \underline{\omega}(A\underline{m} + ai)$, ζ is given by the equation

$$\underline{\omega}(A\underline{m} + ai) = -\zeta^\perp(A\underline{m} + ai) \cdot \zeta^\perp.$$

If $\underline{\omega}(A\underline{m} + ai) = 0$, we take $\zeta \parallel (A\underline{m} + ai)$. Otherwise, we take ($|\underline{\omega}|^2 = -\underline{\omega}_1$)

$$\zeta^\perp = \pm\frac{\underline{\omega}}{|\underline{\omega}|}\frac{A\underline{m} + ai}{|A\underline{m} + ai|}.$$

Finally, we consider $u_k = P(\phi_k\psi, \varphi_k\psi)$ because

$$u_k - \underline{u}h\psi = P(\phi_k\psi, \varphi_k\psi) - P(\phi_k, \varphi_k)\psi = O(k^{-1}),$$

as we wanted. □

Lemma 5.3.2. (H2) *holds for $U = U_A$ given in* (5.17).

We will prove this lemma in section 5.4.1. Now, we check $(\mathrm{H3})_2$ on $\mathscr{D} = \mathbb{T}^2$. To this end, it is convenient to normalize $L^2_S(\mathbb{T}^2; \bar{U}_A)$ by imposing $\int v = 0$ therein.

Proposition 5.3.1. *The space $L^2_S(\mathbb{T}^2; \bar{U}_A)$ is L^2-bounded.*

Proof. Let $u \in L^2_S(\mathbb{T}^2; \bar{U}_A)$. On the one hand, since u is $\bar{U}_A$-valued, we will see in Lemma 5.4.2d that m can be expressed (a.e.) as

$$m = \theta v + \frac{(1-\theta^2)(Av + i)\omega}{1 + \omega\theta A} = \frac{(\theta + \omega A)v + (1-\theta^2)i\omega}{1 + \omega\theta A}, \tag{5.35}$$

for some $\bar{\mathbf{D}}$-valued ω. Hence, by applying

$$\left|\frac{\theta + \omega A}{1 + \omega\theta A}\right|^2 = 1 - (1-\theta^2)\frac{1 - |\omega|^2A^2}{|1 + \omega\theta A|^2} \le 1, \tag{5.36}$$

the triangle inequality yields

$$|m| \le |v| + \frac{1-\theta^2}{1 - |\theta A|} \le |v| + (1 + |\theta|). \tag{5.37}$$

On the other hand, since (5.18)(5.19) is written in the Fourier side as

$$\hat{v}(k)\cdot k = 0, \quad (\hat{v} + A\hat{m} + \hat{\theta}i)(k)\cdot k^\perp = 0, \quad k \in \mathbb{Z}^2,$$

and we have normalized $\hat{v}(0) = 0$, the velocity v is given by

$$\hat{v}(k) = -\frac{k^\perp}{|k|^2}(A\hat{m} + \hat{\theta}i)(k)\cdot k^\perp, \quad k \in \mathbb{Z}^2.$$

Therefore, Plancherel's identity and the triangle inequality yield

$$\|v\|_{L^2} \le \|Am + \theta i\|_{L^2} \le |A|\|m\|_{L^2} + \|\theta\|_{L^2}. \tag{5.38}$$

This concludes the proof since $|\theta| \le 1$ and because (5.37)(5.38) imply

$$\|v\|_{L^2} \le \frac{\|\theta\|_{L^2} + |A|\|1 + |\theta|\|_{L^2}}{1 - |A|}, \qquad \|m\|_{L^2} \le \frac{\|\theta\|_{L^2} + \|1 + |\theta|\|_{L^2}}{1 - |A|}.$$

□

Thus, (H1)-$(\mathrm{H3})_2$ hold on $\mathscr{D} = \mathbb{T}^2$. In order to prove it for $p = \infty$ we need to find bounded U's satisfying (H2). To this end, we will prove the following lemma in section 5.4.2.

Lemma 5.3.3. *For any $R > 0$ there is a bounded open subset U of U_A satisfying* (H2) *and*

$$\{u \in U_A \,:\, |v| < R\} \subset U.$$

Obviously, $(\mathrm{H3})_\infty$ *holds for U.*

Remark 5.3.1. At this point we have all the ingredients to apply the h-principle (Theorems 3.3.1-3.3.3), except we do not know if $L^p_S(\mathscr{D};\bar{U}_A)$ is (weak*) closed (c.f. (H3.2)). Although we have not been able to show it, we have noticed that the proof of this h-principle can be adapted to our $\bar{U}_A$. Recall that the original proof uses this property to show that the set $J^{-1}(0)$ consists of functions u solving (L_A, K). Here, we overcome this obstacle by checking that the residual subset X_J of $J^{-1}(0)$ satisfies this requirement.

Proof of Theorem 5.2.1. Let $\bar{u} \in C([0,T]; L^p_S(\mathscr{D};\bar{U}_A))$ be a strict subsolution to IPM_A w.r.t. Ω_{mix}. For $p = 2$ we take $U = U_A$ and for $p = \infty$ we take U from Lemma 5.3.3 in such a way that $|\bar{v}| < R$.

Let us recall how X_0 is defined in section 3.3. A subsolution $u \in C([0,T]; L^p_A(\mathscr{D};\bar{U}_A))$ belongs to $X_0 \equiv X_0(\bar{u},\mathcal{F})$ if

$$u = \bar{u} \quad \text{outside} \quad \Omega_{\mathrm{mix}},$$

and it is perturbable inside

$$u \in C(\Omega_{\mathrm{mix}}; U_A).$$

In addition, we ask u to satisfy the following property. There is $c(u) \in (0,1)$ so that, at each $t \in [0,T]$, for $F = \mathrm{id}$ and P,

$$\left| \fint_R [F(u) - F(\bar{u})](t,x)\,\mathrm{d}x \right| \leq cE(t,R),$$

for every bounded rectangle $\emptyset \neq R \subset \Omega_{\mathrm{mix}}(t)$. By $(\mathrm{H3})_p$, the closure X of X_0 in $C([0,T]; L^p_S(\mathscr{D}))$ is a completely metrizable space.

Given $I \times \Omega \subset\subset \Omega_{\mathrm{mix}}$ with $I = [t_1, t_2]$ and Ω open, the relaxation-error is defined in section 3.4 as the functional

$$\begin{aligned} J : X \;&\rightarrow\; \mathbb{R}_+ \\ u \;&\mapsto\; \sup_{t\in I} \int_\Omega (1 - \theta(t,x)^2)\,\mathrm{d}x, \end{aligned}$$

which is well defined because, by convexity, $|\theta| \leq 1$ for states in X. Indeed, J is upper-semicontinuous, and so the set X_J of continuity points of J is countable intersection of open dense sets. Then, following section 3.4, the hypothesis (H1)-$(\mathrm{H3})_p$ imply that $X_J \subset J^{-1}(0)$. In contrast to section 5.2, here we cannot use that $L^p_S(\mathscr{D};\bar{U}_A)$ is (weak*) closed to ensure that the functions in $J^{-1}(0)$ are K-valued in $I \times \Omega$. However, we will prove that X_J satisfies this requirement.

Given $u \in X_J$ let $(u_k) \subset X_0$ converging to u. Fix $t \in I$. We claim that $\theta_k(t) \to \theta(t)$ in $L^q(\Omega)$ for every $1 < q < \infty$. Indeed, since $J(u) = 0$ and

$$|\|\theta(t)\|^q_{L^q} - \|\theta_k(t)\|^q_{L^q}| = \int_\Omega (1 - |\theta_k(t,x)|^q)\,\mathrm{d}x \leq C_q \int_\Omega (1 - \theta_k(t,x)^2)\,\mathrm{d}x \leq C_q J(u_k) \to C_q J(u) = 0,$$

the claim follows by convexity. Now take $1 < q < p$ and denote $f = (m - \theta v)$, $f_k = (m_k - \theta v_k)$ and $\tilde{f}_k = (m_k - \theta_k v_k)$. On the one hand, by convexity and applying $f_k(t) \overset{*}{\rightharpoonup} f(t)$, we get

$$\int_\Omega |m - \theta v|^q(t,x)\,\mathrm{d}x = \|f(t)\|_{L^q}^q \leq \liminf_k \|f_k(t)\|_{L^q}^q = \liminf_k \int_\Omega |m_k - \theta v_k|^q(t,x)\,\mathrm{d}x. \tag{5.39}$$

On the other hand, by applying the inverse triangle inequality, we obtain

$$|\|f_k(t)\|_{L^q} - \|\tilde{f}_k(t)\|_{L^q}|^q \leq \|f_k(t) - \tilde{f}_k(t)\|_{L^q}^q = \int_\Omega |\theta - \theta_k|^q |v_k|^q(t,x)\,\mathrm{d}x \to 0, \tag{5.40}$$

where the last convergence follows from Hölder's inequality and $(\mathrm{H3})_p$. Finally, by applying (5.39)(5.40) and that u_k is $\bar{U}_A$-valued (see (5.17)), we deduce

$$\begin{aligned}
\int_\Omega |m - \theta v|^q(t,x)\,\mathrm{d}x &\leq \liminf_k \int_\Omega |m_k - \theta v_k|^q(t,x)\,\mathrm{d}x \\
&= \liminf_k \int_\Omega |m_k - \theta_k v_k|^q(t,x)\,\mathrm{d}x \\
&\leq \liminf_k \int_\Omega (1 - (\theta_k)^2)^q \frac{|Av_k + i|^q}{(1 - \theta_k A)^q}(t,x)\,\mathrm{d}x = 0,
\end{aligned}$$

and so $m = \theta v$. Therefore, $u(t)$ is K-valued on Ω. The rest follows as in section 3.4. $\square$

5.4 The relaxation

First of all let us recall several concepts given in section 3.2. Given a set K and a cone Λ in $\mathbb{R}^N$, the Λ-lamination of order 1 of K is

$$K^{1,\Lambda} := \{\tfrac{1+s}{2}u_1 + \tfrac{1-s}{2}u_2 \,:\, s \in [-1,1],\, u_1, u_2 \in K \text{ s.t. } u_1 - u_2 \in \Lambda\}, \tag{5.41}$$

and, inductively, the Λ-lamination of order $n \geq 2$ of K is

$$K^{n,\Lambda} := (K^{n-1,\Lambda})^{1,\Lambda}.$$

This generates an ascending chain of sets $K \subset K^{1,\Lambda} \subset K^{2,\Lambda} \subset \cdots$ whose limit $K^{lc,\Lambda} := \bigcup K^{n,\Lambda}$ is the Λ-**lamination hull** of K. This is contained in the Λ-**convex hull** of K which is defined as follows: A state $u \in \mathbb{R}^N$ does not belong to K^Λ if there is a Λ-convex function f (meaning that $\lambda \mapsto f(u_0 + \lambda \underline{u})$ is convex for all $u_0 \in \mathbb{R}^N$ and $\underline{u} \in \Lambda$) so that $f \leq 0$ on K and $f(u) > 0$.

From now on we consider K and Λ_A given in (5.12) and (5.32) respectively. In order to alleviate the notation we shall omit the tag "A" wherever we do not need to distinguish between the cases $A = 0$ and $A \neq 0$. Thus, we shall abbreviate $L \equiv L_A$, $\Lambda \equiv \Lambda_A$ and $U \equiv U_A$.

This section is split in three parts. Firstly we compute $K^{1,\Lambda}$ since it contains the key to understand the relaxation. Secondly we prove Lemmas 5.3.2 (sec. 5.4.1) and 5.3.3 (sec. 5.4.2). Finally we check that $K^{lc,\Lambda} = \bar{U}$ and $(K_M)^{lc,\Lambda} = \bar{U}_M$ (sec. 5.4.3).

Lemma 5.4.1. *Let $u = (\theta, v, m) \in [-1,1] \times \mathbb{R}^2 \times \mathbb{R}^2$. The following are equivalent:*

(a) $u \in K^{1,\Lambda}$.

(b) $u \in K^{1,\Lambda_1}$.

(c) There are $(\underline{m},\underline{\omega}) \in \mathbb{R}^2 \times \mathbf{S}$ so that

$$v = \underline{m} + \theta\underline{v}, \qquad m = \underline{v} + \theta\underline{m},$$

where $\underline{v} = \underline{\omega}(A\underline{m} + i)$, or equivalently,

$$v = L_{\theta\underline{\omega}}(\underline{m}) := \underline{m} + \theta\underline{\omega}(A\underline{m} + i), \qquad m = \theta v + (1-\theta^2)(A\underline{m} + i)\underline{\omega}.$$

(d) There is $\underline{\omega} \in \mathbf{S}$ so that

$$(1 + \underline{\omega}\theta A)(m - \theta v) = (1-\theta^2)(Av + i)\underline{\omega}.$$

(e) $u \in \partial U$, that is,

$$|2(1-\theta A)(m-\theta v) + (1-\theta^2)(Av+i)| = (1-\theta^2)|Av+i|.$$

(f) $f(u) = 0$, where

$$f(u) := |2(1-\theta A)(m-\theta v) + (1-\theta^2)(Av+i)| - (1-\theta^2)|Av+i|.$$

(g) $g(u) = 0$, where

$$\begin{aligned} g(u) &:= ((1-\theta A)(m-\theta v) + (1-\theta^2)(Av+i)) \cdot (m-\theta v) \\ &= (m + Av + i - \theta(v + Am + \theta i)) \cdot (m - \theta v) \\ &= (m + Av + i) \cdot (m - \theta v) - \theta(v + Am + \theta i) \cdot m - \theta \det L_A(u). \end{aligned}$$

Proof. By definition (5.41) a state $u = (\theta, v, m)$ belongs to $K^{1,\Lambda}$ if and only if there are $s \in [-1,1]$, $u_1, u_2 \in K$ so that $u_1 - u_2 \in \Lambda$ and

$$u = \tfrac{1+s}{2}u_1 + \tfrac{1-s}{2}u_2 = \langle u \rangle + s\underline{u}, \tag{5.42}$$

where $\langle u \rangle \equiv \frac{u_1+u_2}{2}$ and $\underline{u} \equiv \frac{u_1-u_2}{2}$. Since $u_j \in K$, we have $|\theta_j| = 1$ and $m_j = \theta_j v_j$ for $j = 1,2$.

$a \Leftrightarrow b$: Let us assume that $\underline{\theta} = 0$ ($\underline{u} \in \Lambda_0$). On the one hand, $\theta_1 = \theta_2 = \theta$. Hence, $m_j = \theta v_j$ for $j = 1,2$, and so $\underline{m} = \theta\underline{v}$. On the other hand, $\underline{v} = -A\underline{m}$. Thus, necessarily $\underline{u} = 0$ ($u_1 = u_2$). Therefore, $K^{1,\Lambda_0} = K$.

$b \Leftrightarrow c$: Now let us assume that $\underline{\theta} \neq 0$ ($\underline{u} \in \Lambda_1$). On the one hand, w.l.o.g. (relabelling if necessary) we may assume that $\theta_1 = -\theta_2 = 1$. Hence $m_1 = v_1$ and $m_2 = -v_2$, and so $\langle m \rangle = \underline{v}$ and $\underline{m} = \langle v \rangle$. Thus, (5.42) reads as

$$(\theta, v, m) = (0, \underline{m}, \underline{v}) + s(1, \underline{v}, \underline{m}). \tag{5.43}$$

On the other hand, there is $\underline{\omega} \in \mathbf{S}$ so that $\underline{v} = \underline{\omega}(A\underline{m} + i)$. Thus, (5.43) reads as

$$\begin{aligned} \theta &= s, \\ v &= \underline{m} + \theta\underline{v} = L_{\theta\underline{\omega}}(\underline{m}), \\ m &= \underline{v} + \theta\underline{m} = \theta v + (1-\theta^2)(A\underline{m} + i)\underline{\omega}. \end{aligned}$$

$c \Leftrightarrow d$: By definition, the map $L_{\theta\underline{\omega}}$ satisfies the identity

$$AL_{\theta\underline{\omega}}(\underline{m}) + i = (1 + \underline{\omega}\theta A)(A\underline{m} + i). \tag{5.44}$$

This concludes the proof because $\underline{m} = L_{\theta\underline{\omega}}^{-1}(v)$ and $(1 + \underline{\omega}\theta A) \neq 0$.

$d \Leftrightarrow e$: Although this equivalence can be checked directly by elementary computations, let us give a shorter geometric proof. For any $b \in \mathbb{D}$ let us consider the automorphism of the shifted disc $\bar{\mathbf{S}}$

$$\varphi_b(\omega) := \frac{(1-b)\omega}{1+\omega b}. \tag{5.45}$$

This can be expressed in terms of the classical automorphism of the unit disc $\mathbb{D}$ (recall (5.33))

$$\tilde{\varphi}_a(u) := \frac{u-a}{1-a^*u},$$

as $\varphi_b(\omega) = T^{-1}\tilde{\varphi}_{a(b)}(T\omega)$ where $a(b) = \frac{b}{2-b} \in \mathbb{D}$. From Complex Analysis it is well known that $\varphi_b \in \mathrm{Aut}(\mathbf{S})$ and also $\varphi_b \in \mathrm{Aut}(\mathbf{D})$. Thus, d reads as

$$(1 - \theta A)(m - \theta A) = (1 - \theta^2)(Av + i)\varphi_{\theta A}(\underline{\omega}).$$

This concludes the proof since $\varphi_{\theta A} \in \mathrm{Aut}(\mathbf{S})$.

$e \Leftrightarrow f$: Trivial.

$f \Leftrightarrow g$: This follows from

$$4(1-\theta A)g(u) = |2(1-\theta A)(m-\theta v) + (1-\theta^2)(Av+i)|^2 - (1-\theta^2)^2|Av+i|^2, \tag{5.46}$$

and the fact that $(1 - \theta A) > 0$. $\square$

Lemma 5.4.2. *Let $u = (\theta, v, m) \in (-1, 1) \times \mathbb{R}^2_A \times \mathbb{R}^2$ where $\mathbb{R}^2_A := \{v \in \mathbb{R}^2 : Av + i \neq 0\}$. The following are equivalent:*

(d) There is $\omega \in \mathbf{D}$ so that

$$(1 + \omega\theta A)(m - \theta v) = (1 - \theta^2)(Av + i)\omega.$$

(e) $u \in U$, that is,

$$|2(1 - \theta A)(m - \theta v) + (1 - \theta^2)(Av + i)| < (1 - \theta^2)|Av + i|.$$

(f) $f(u) < 0$.

(g) $g(u) < 0$.

Proof. $d \Leftrightarrow e$: Analogously to the proof of the equivalence $d \Leftrightarrow e$ in Lemma 5.4.1, this follows from the fact that $\varphi_{\theta A} \in \mathrm{Aut}(\mathbf{D})$. $e \Leftrightarrow f$: Trivial. $f \Leftrightarrow g$: This follows from (5.46) and $(1 - \theta A) > 0$. $\square$

Remark 5.4.1. The equivalences $d \Leftrightarrow e$ are trivial for $A = 0$ because $\varphi_0 = \mathrm{id}$ (cf. (5.45)). For a general $|A| < 1$, U_A can be understood as $(-1,1) \times \mathbb{R}^2_A \times \mathbf{D}$ via the change of variables

$$\begin{aligned} U_A &\simeq (-1,1) \times \mathbb{R}^2_A \times \mathbf{D} \\ (\theta, v, m) &\leftrightarrow (\theta, v, \omega) \end{aligned}$$

given by

$$m = \theta v + (1-\theta^2)(Av + i) \bigcirc \quad \text{where} \quad \bigcirc = \underbrace{\frac{\omega}{1 + \omega\theta A}}_{(d)} = \underbrace{\frac{\varphi_{\theta A}(\omega)}{1 - \theta A}}_{(e)}.$$

Thus, given $u \in U_A$ near to some $u_0 \in K^{1,\Lambda_A} = \partial U_A$, while $\omega \in \mathbf{D}$ is near to the direction $\underline{\omega}(u_0) \in \mathbf{S} = \partial \mathbf{D}$ (coupled with $\underline{m} = L^{-1}_{\theta\underline{\omega}}(v)$) used to construct u_0 in Lemma 5.4.1d, the transformation $\varphi_{\theta A}(\omega)$ represents the position of m in the ball defined by Lemma 5.4.2e.

5.4.1 Proof of Lemma 5.3.2

This follows from the below stronger version of Lemma 5.3.2.

Lemma 5.4.3. *There is $d_A > 0$ such that, for all $u \in U$ there is $\underline{u} \in \Lambda$ with $\underline{\theta} = 1$ for which*

$$u + \lambda \underline{u} \in U, \quad |\lambda| \le d_A(1 - \theta^2).$$

Proof. Given $u = (\theta, v, m) \in U$, let $\underline{u} = (1, \underline{v}, \underline{m}) \in \Lambda_1$, that is, $\underline{v} = \underline{\omega}(A\underline{m} + i)$ for some $(\underline{m}, \underline{\omega}) \in \mathbb{R}^2 \times \mathbf{S}$ to be determined. Since U is open, there is $\epsilon(u, \underline{u}, U) > 0$ so that $u_\lambda \equiv u + \lambda\underline{u} \in U$ for all $|\lambda| \le \epsilon$, that is,

$$|\theta_\lambda| < 1, \qquad Av_\lambda + i \neq 0, \tag{5.47}$$

and there is $\omega_\lambda \in \mathbf{D}$ satisfying (Lemma 5.4.2d)

$$(1 + \omega_\lambda\theta_\lambda A)(m_\lambda - \theta_\lambda v_\lambda) = (1 - (\theta_\lambda)^2)(Av_\lambda + i)\omega_\lambda, \tag{5.48}$$

for all $|\lambda| \le \epsilon$. To prove Lemma 5.4.3 we must find some $\underline{u}$ making ϵ big enough, namely $\epsilon(1 - \theta^2, A)$. Roughly speaking, if u is far from ∂U, ϵ is controlled easily. Conversely, if u is close to ∂U, a priori ϵ is comparable to $\mathrm{dist}(u, \partial U)$, unless we take $\underline{u}$ somehow "parallel" to ∂U. In light of Remark 5.4.1, it seems suitable to consider $\underline{m} = L^{-1}_{\theta\underline{\omega}}(v)$ with $\underline{\omega} \approx \omega_0$ to be determined. Let us see that this choice works. We split the proof in two steps. Firstly (*step 1*) we prove the statement by assuming a claim. Secondly (*step 2*) this claim is proved by elementary computations.

Step 1. Claim: Let us take $\underline{m} = L^{-1}_{\theta\underline{\omega}}(v)$ with $\underline{\omega} \in \mathbf{S}$ to be determined. Then, (5.47) holds for all $|\lambda| \le \frac{1}{2}(1 - \theta^2)$ and (5.48) is equivalent to

$$\lambda\alpha|T\underline{\omega} - T\omega| < (1 - \theta^2)(1 - |T\omega|^2), \tag{5.49}$$

where $\omega \equiv \omega_0$ and $|\alpha| \le \alpha_A$ for some constant $\alpha_A > 0$. We shall prove this claim in the *step 2*. Assume that this claim is true. Hence, if we make the change of variables $\lambda = d(1 - \theta^2)$ for $d \in \mathbb{R}$, (5.49) reads as

$$d\alpha|T\underline{\omega} - T\omega| < (1 - |T\omega|^2). \tag{5.50}$$

If $|T\omega| \le \frac{1}{2}$ (u is far from ∂U) we take $\underline{\omega} = 0 \in \mathbf{S}$ and then (5.50) holds for every $|d| \le \frac{1}{2\alpha_A}$.
If $\frac{1}{2} < |T\omega| < 1$ (u is close to ∂U) we take $T\underline{\omega} = \frac{T\omega}{|T\omega|}$ and then (5.50) reads as

$$d\alpha < (1 + |T\omega|),$$

which holds for every $|d| \le \frac{1}{\alpha_A}$. Therefore, we can take $d_A = \frac{1}{2\alpha_A}$.

Step 2. Proof of the claim: Since $\underline{v} = \underline{\omega}(A\underline{m} + i)$ and $\underline{m} = L^{-1}_{\theta\underline{\omega}}(v)$, Lemma 5.4.1c and (5.44) yield

$$\underline{v} = \omega \frac{Av + i}{1 + \underline{\omega}\theta A}, \qquad \underline{m} = v - \theta\underline{v}.$$

Let us expand the factors of (5.48) in terms of λ. They are

$$m_\lambda - \theta_\lambda v_\lambda = (m - \theta v) + \lambda(\underline{m} - (\theta\underline{v} + v)) - \lambda^2\underline{v} = (m - \theta v) + (\theta^2 - (\theta_\lambda)^2)\underline{v}, \tag{5.51a}$$

$$Av_\lambda + i = (Av + i) + \lambda\underline{v} = (Av + i)\frac{1 + \underline{\omega}\theta_\lambda A}{1 + \underline{\omega}\theta A}. \tag{5.51b}$$

Since $u \in U$, we have $|\theta| < 1$ and $Av + i \ne 0$. Then, by (5.51b): $|\theta_\lambda| < 1 \Rightarrow Av_\lambda + i \ne 0$. Therefore, (5.47) is equivalent to $|\theta + \lambda| < 1$, and this holds for all $|\lambda| \le \frac{1}{2}(1 - \theta^2)$.
By (5.51), if we multiply (5.48) by $(1 + \underline{\omega}\theta A)(1 + \omega\theta A)/(Av + i)$, we get

$$\begin{aligned}(1 + \omega_\lambda\theta_\lambda A)((1 + \underline{\omega}\theta A)(1 - \theta^2)\omega + (1 + \omega\theta A)(\theta^2 - (\theta_\lambda)^2)\underline{\omega})\\ = (1 - (\theta_\lambda)^2)(1 + \omega\theta A)(1 + \underline{\omega}\theta_\lambda A)\omega_\lambda.\end{aligned} \tag{5.52}$$

Hence, by applying the following identities

$$\begin{aligned}(1 + \underline{\omega}\theta A)(1 - \theta^2)\omega + (1 + \omega\theta A)(\theta^2 - (\theta_\lambda)^2)\underline{\omega} = (1 - \theta^2)(\omega - \underline{\omega}) + (1 + \omega\theta A)(1 - (\theta_\lambda)^2)\underline{\omega},\\ (1 + \underline{\omega}\theta_\lambda A)\omega_\lambda = (\omega_\lambda - \underline{\omega}) + (1 + \omega_\lambda\theta_\lambda A)\underline{\omega},\end{aligned}$$

(5.52) reads as

$$(1 - \theta^2)(1 + \omega_\lambda\theta_\lambda A)(\underline{\omega} - \omega) = (1 - (\theta_\lambda)^2)(1 + \omega\theta A)(\underline{\omega} - \omega_\lambda). \tag{5.53}$$

Since (recall (5.33)) $w = \frac{1}{2}(Tw - 1)$ for all $w \in \mathbb{R}^2$, (5.53) reads as

$$(1 - \theta^2)((2 - \theta_\lambda A) + \theta_\lambda AT\omega_\lambda)(T\underline{\omega} - T\omega) = (1 - (\theta_\lambda)^2)((2 - \theta A) + \theta AT\omega)(T\underline{\omega} - T\omega_\lambda),$$

or equivalently, $\zeta T\omega_\lambda = \eta$ where we have abbreviated

$$\begin{aligned}\zeta &\equiv (1 - (\theta_\lambda)^2)((2 - \theta A) + \theta AT\omega) + (1 - \theta^2)\theta_\lambda A(T\underline{\omega} - T\omega),\\ \eta &\equiv (1 - (\theta_\lambda)^2)((2 - \theta A) + \theta AT\omega)T\underline{\omega} - (1 - \theta^2)(2 - \theta_\lambda A)(T\underline{\omega} - T\omega).\end{aligned}$$

In this way: $|\eta| < |\zeta| \Rightarrow \omega_\lambda \in \mathbf{D}$. Let us write the inequality $|\eta|^2 < |\zeta|^2$. Since $|T\underline{\omega}| = 1$, the term $(1 - (\theta_\lambda)^2)^2|(2 - \theta A) + \theta AT\omega|^2$ is cancelled. Hence, by reordering the remainder terms, the inequality $|\eta|^2 < |\zeta|^2$ is equivalent to

$$\begin{aligned}(1 - \theta^2)((2 - \theta_\lambda A)^2 - (\theta_\lambda A)^2)|T\underline{\omega} - T\omega|^2\\ < 2(1 - (\theta_\lambda)^2)(((2 - \theta A) + \theta AT\omega)((2 - \theta_\lambda A)T\underline{\omega} + \theta_\lambda A)) \cdot (T\underline{\omega} - T\omega),\end{aligned} \tag{5.54}$$

where we have eliminated a factor $(1-\theta^2)>0$. Notice that (5.54) can be written as $p(\lambda)<0$ for some (3-degree) polynomial p in λ. In particular, (5.54) can be written as

$$\lambda\left(\int_0^1 \partial_\lambda q(s\lambda)\,\mathrm{d}s\right)\cdot(T\underline{\omega}-T\omega)<-p(0), \tag{5.55}$$

where $p(\lambda)=q(\lambda)\cdot(T\underline{\omega}-T\omega)$, that is,

$$\begin{aligned} q(\lambda)\equiv\,&(1-\theta^2)((2-\theta_\lambda A)^2-(\theta_\lambda A)^2)(T\underline{\omega}-T\omega)\\ &-2(1-(\theta_\lambda)^2)((2-\theta A)+\theta AT\omega)((2-\theta_\lambda A)T\underline{\omega}+\theta_\lambda A). \end{aligned}$$

On the one hand, since $|\lambda|,|\theta|,|A|,|T\omega|,|T\underline{\omega}|\leq 1$ we can bound

$$\left|\int_0^1 \partial_\lambda q(s\lambda)\,\mathrm{d}s\right|\leq C, \tag{5.56}$$

for some constant $C>0$. On the other hand, $-p(0)=(1-\theta^2)\beta$ where we have abbreviated

$$\beta\equiv 2(((2-\theta A)+\theta AT\omega)((2-\theta A)T\underline{\omega}+\theta A))\cdot(T\underline{\omega}-T\omega)-((2-\theta A)^2-(\theta A)^2)|T\underline{\omega}-T\omega|^2.$$

Remarkably, using $|T\underline{\omega}|=1$ and abbreviating $a\equiv\frac{\theta A}{2-\theta A}$, this term can be greatly simplified

$$\begin{aligned} \beta&=(2-\theta A)^2(2((1+aT\omega)(T\underline{\omega}+a))\cdot(T\underline{\omega}-T\omega)-(1-a^2)|T\underline{\omega}-T\omega|^2)\\ &=(2-\theta A)^2((1+a^2)(T\underline{\omega}+T\omega)+2a(1+T\underline{\omega}T\omega))\cdot(T\underline{\omega}-T\omega)\\ &=(2-\theta A)^2|1+aT\underline{\omega}|^2(1-|T\omega|^2)\\ &=4|1+\underline{\omega}\theta A|^2(1-|T\omega|^2). \end{aligned} \tag{5.57}$$

By applying (5.56)(5.57) on (5.55), we deduce (5.49) with

$$\alpha\equiv\frac{1}{4|1+\underline{\omega}\theta A|^2}\left(\int_0^1 \partial_\lambda q(s\lambda)\,\mathrm{d}s\right)\cdot\frac{(T\underline{\omega}-T\omega)}{|T\underline{\omega}-T\omega|},$$

which satisfies $|\alpha|\leq\frac{C}{4(1-|A|)^2}$. □

5.4.2 Proof of Lemma 5.3.3

As in [126], the relaxed set U is unbounded, thereby preventing from constructing L^∞-solutions from the h-principle applied to U-valued subsolutions. In order to find bounded subsets of U satisfying (H2) we have to restrict K somehow. In [126] ($A=0$) Székelyhidi computed explicitly the Λ_0-convex hull of

$$K_M:=\{u\in K\,:\,|\underbrace{2v+\theta i}_{\equiv\mathbf{v}}|\leq M\}=\{u\in K\,:\,\underbrace{4v\cdot(v+\theta i)}_{\Lambda_0\text{-linear}}\leq M^2-1\},$$

for any $M>1$ (notice $K_M\subset\subset K$) which is given by the following 4 inequalities:

$$|2(m-\theta v)+(1-\theta^2)i|<(1-\theta^2), \tag{5.58a}$$

$$4v\cdot(v+\theta i)<M^2-1, \tag{5.58b}$$

$$|2(m-v)+(1-\theta)i|<M(1-\theta), \tag{5.58c}$$

$$|2(m+v)+(1+\theta)i|<M(1+\theta). \tag{5.58d}$$

As observed in [126], these inequalities are linked by the following identity:

$$
\begin{aligned}
&(5.59a) && (1-\theta^2)^2 - |2(m-\theta v) + (1-\theta^2)i|^2 \\
&(5.59b) && + (1-\theta^2)(M^2 - 1 - 4v\cdot(v+\theta i)) \\
&(5.59c) && = \frac{1+\theta}{2}(M^2(1-\theta)^2 - |2(m-v) + (1-\theta)i|^2) \\
&(5.59d) && + \frac{1-\theta}{2}(M^2(1+\theta)^2 - |2(m+v) + (1+\theta)i|^2),
\end{aligned}
$$

which is indeed crucial to prove (H2).

Remark 5.4.2. In [126] Székelyhidi introduces the smart (linear) change of variables $(\theta, \mathbf{v}, \mathbf{n}) = (\theta, 2v + \theta i, 2m + i)$, which simplifies significantly the computations and inequalities in (5.58). Under this transformation: 1) the wave cone reads as $\Lambda_0 = \{\underline{u} \in \mathbb{R}^5 : |\underline{\theta}| = |\underline{\mathbf{v}}|\}$ because (5.3)(5.5) become symmetric, 2) the geometry of K is preserved (given $|\theta| = 1$: $m = \theta v \Leftrightarrow \mathbf{n} = \theta\mathbf{v}$). After this, Székelyhidi computed the Λ_0-convex hull of $K_M = \{u \in K : |\mathbf{v}| \leq M\}$.

For a general $|A| < 1$, the corresponding change of variables that keeps 1) and 2) is $(\theta, \mathbf{v}, \mathbf{n}) = (\theta, 2v + Am + \theta i, (2+\theta A)m + i)$, which is not linear in $\mathbf{n}$ for $A \neq 0$, thereby hampering the plane wave analysis. Thus, for $A \neq 0$, although $\mathbf{v} = 2v + Am + \theta i$ symmetrizes (5.3)(5.5), any linear change of variables in $\mathbf{n}$ messes the simplicity of K up. This is why we have chosen not to make a change of variables in this case.

In this regard, for $A \neq 0$ it is not evident what restriction of K may return a simple Λ_A-convex hull as in (5.58). To overcome this drawback, inspired by (5.59), instead of restricting K first, we start trying to extend properly the identity (5.59) to $|A| < 1$, with the hope that this will reveal the analogous inequalities to (5.58) that describe the Λ_A-convex hull of some restriction of K. Fortunately, this is the case.

Lemma 5.4.4. *For every $M \in \mathbb{R}$ and $u = (\theta, v, m) \in [-1,1] \times \mathbb{R}^2 \times \mathbb{R}^2$,*

$$
\begin{aligned}
&(5.60a) && \frac{1}{1-\theta A}((1-\theta^2)^2|Av+i|^2 - |2(1-\theta A)(m-\theta v) + (1-\theta^2)(Av+i)|^2) \\
&(5.60b) && + (1-\theta^2)(M^2 - 1 - 4v\cdot(v + Am + \theta i + Ai)) \\
&(5.60c) && = \frac{1+\theta}{2}((M^2 - A)(1-\theta)^2 - (1-A)|2(m-v) + (1-\theta)i|^2) \\
&(5.60d) && + \frac{1-\theta}{2}((M^2 + A)(1+\theta)^2 - (1+A)|2(m+v) + (1+\theta)i|^2).
\end{aligned}
$$

Proof. First notice that, by (5.46), we have (5.60a) $= -4g(u)$. On the one hand,

$$
\begin{aligned}
(5.60a) + (5.60b) &= 4(\theta(v + Am + \theta i) - (m + Av + i))\cdot m + 4\theta(m + Av + i)\cdot v \\
&\quad + (1-\theta^2)(M^2 - 1 - 4Av_2) - 4v\cdot(v + Am + \theta i) \\
&= -4(1-\theta A)(|m|^2 + |v|^2) - 8(A-\theta)m\cdot v \\
&\quad + (1-\theta^2)(M^2 - 1 - 4(m_2 + Av_2)).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
(5.60c) + (5.60d) &= \frac{1+\theta}{2}((M^2-1)(1-\theta)^2 - 4(1-A)(|m-v|^2 + (1-\theta)(m_2-v_2))) \\
&+ \frac{1-\theta}{2}((M^2-1)(1+\theta)^2 - 4(1+A)(|m+v|^2 + (1+\theta)(m_2+v_2))) \\
&= -2((1+\theta)(1-A)|m-v|^2 + (1-\theta)(1+A)|m-v|^2) \\
&+ (1-\theta^2)(M^2 - 1 - 2(1-A)(m_2-v_2) - 2(1+A)(m_2+v_2)).
\end{aligned}
$$

This concludes the proof. □

Observe that (5.60) generalizes (5.59). For any $M > 1$, we consider the open set $U_{A,M}$ of states $u \in [-1,1] \times \mathbb{R}^2 \times \mathbb{R}^2$ given by the following 4 inequalities:

$$
|2(1-\theta A)(m-\theta v) + (1-\theta^2)(Av+i)| < (1-\theta^2)|Av+i|, \tag{5.61a}
$$

$$
4v \cdot (v + Am + \theta i + Ai) < M^2 - 1, \tag{5.61b}
$$

$$
|2(m-v) + (1-\theta)i| < M_{-A}(1-\theta), \tag{5.61c}
$$

$$
|2(m+v) + (1+\theta)i| < M_{+A}(1+\theta), \tag{5.61d}
$$

where

$$
M_{\pm A} \equiv \sqrt{\frac{M^2 \pm A}{1 \pm A}}.
$$

By analogy with [126], (5.60) suggests that $U_{A,M}$ is the interior of the Λ_A-convex hull of

$$
K_{A,M} := \{u \in K \,:\, |2v + \theta i| \le M_{\theta A}\} = \{u \in K \,:\, B_A(u) \le M^2 - 1\},
$$

where we have abbreviated

$$
B_A(u) := |\mathbf{b}_A(u) + v|^2 - |\mathbf{b}_A(u) - v|^2 = 4v \cdot \mathbf{b}_A(u), \tag{5.62}
$$

and

$$
\mathbf{b}_A(u) := v + Am + \theta i + Ai.
$$

Observe that $K_{0,M} = K_M$. In section 5.4.3 we shall prove that both $K^{lc,\Lambda_A} = \bar{U}_A$ and $(K_{A,M})^{lc,\Lambda_A} = \bar{U}_{A,M}$. Now, let us continue with the proof of Lemma 5.3.3. Thus, from now on we shall omit the tag "A" wherever we do not need to distinguish between the cases $A = 0$ and $A \neq 0$.

Firstly, let us check that U_M is indeed bounded.

Lemma 5.4.5. *Let $M > 1$. The set U_M is bounded.*

Proof. Given $u \in U$ there is $\omega \in \mathbf{D}$ so that (5.35) holds. In particular,

$$
Am + \theta i = \frac{\theta + \omega A}{1 + \omega\theta A}(Av + i).
$$

Then, by applying (5.36), we have $|Am + \theta i| \le |Av + i| \le |A||v| + 1$. Hence, (5.61b)(5.62) imply

$$
4|v|^2 = B(u) - 4v \cdot (Am + \theta i + Ai) < M^2 - 1 + 4|v|(|A||v| + 1 + |A|),
$$

and so

$$4((1-|A|)|v|-(1+|A|))|v|<M^2-1.$$

Thus, necessarily

$$|v|<\frac{(1+|A|)+\sqrt{(1+|A|)^2+(1-|A|)(M^2-1)}}{2(1-|A|)}.$$

Finally, recall that m is controlled by (5.37). □

Secondly, let us show that these U_M's contain simpler sets as stated in Lemma 5.3.3.

Lemma 5.4.6. *For any $R>0$ there is $M>1$ so that*

$$\{u\in U\,:\,|v|<R\}\subset U_M.$$

Proof. Let $u=(\theta,v,m)\in U$ with $|v|<R$. By Lemma 5.4.2d, there is $\omega\in\mathbf{D}$ so that

$$m=\theta v+(1-\theta^2)\frac{(Av+i)\omega}{1+\omega\theta A}.$$

Thus, for (5.61c)(5.61d) we have

$$|2(m\pm v)+(1\pm\theta)i|=(1\pm\theta)\left|\pm 2v+i+(1\mp\theta)\frac{(Av+i)\omega}{1+\omega\theta A}\right|\le(1\pm\theta)C_\pm,$$

for some constant $C_\pm(A,R)>0$. Concerning (5.61b) we have

$$1+B(u)\le C,$$

for some constant $C(A,R)>0$. Hence, since there is $M(A,R)>1$ satisfying $C_\pm\le M_\pm$ and $C\le M^2$, we have $u\in U_M$. □

Finally, the following lemma completes the proof of Lemma 5.3.3.

Remark 5.4.3. The pinch singularity $Av+i=0$ becomes further complicated for $U_{A,M}$ because the new inequalities (5.61b)-(5.61d) can interfere with it for the particular value (cf. 5.64)

$$M_*(A):=\sqrt{1+4\left(\frac{1}{A^2}-1\right)}. \tag{5.63}$$

Notice that M_* is symmetric and strictly decreasing on $(0,1]$ with $M_*(0)=+\infty$ and $M_*(1)=1$. For simplicity we shall omit this case.

Lemma 5.4.7. *Let $1<M\neq M_*(A)$. The set U_M satisfies* (H2).

Proof. Given $(\theta,v)\in(-1,1)\times\mathbb{R}^2$ we consider the subsets of $\mathbb{R}^2$

$$\begin{aligned}
\mathbb{B}(\theta,v)&:=\{m\in\mathbb{R}^2\,:\,|2(1-\theta A)(m-\theta v)+(1-\theta^2)(Av+i)|<(1-\theta^2)|Av+i|\},\\
\mathbb{H}(\theta,v)&:=\{m\in\mathbb{R}^2\,:\,4v\cdot(v+Am+\theta i+Ai)<M^2-1\},\\
\mathbb{B}_-(\theta,v)&:=\{m\in\mathbb{R}^2\,:\,|2(m-v)+(1-\theta)i|<M_-(1-\theta)\},\\
\mathbb{B}_+(\theta,v)&:=\{m\in\mathbb{R}^2\,:\,|2(m+v)+(1+\theta)i|<M_+(1+\theta)\}.
\end{aligned}$$

By definition, a state $u = (\theta, v, m) \in (-1, 1) \times \mathbb{R}^2 \times \mathbb{R}^2$ belongs to U if and only if m belongs to the open ball $U(\theta, v) := \mathbb{B}(\theta, v)$. Similarly, u belongs to the bounded subset U_M if and only if m belongs to $U_M(\theta, v) := (\mathbb{B} \cap \mathbb{H} \cap \mathbb{B}_- \cap \mathbb{B}_+)(\theta, v)$. Notice that $\mathbb{B}_-(\theta, v)$ and $\mathbb{B}_+(\theta, v)$ are (open) balls. The geometry of $\mathbb{H}_A(\theta, v)$ depends on A (cf. Fig. 5.1). On the one hand, for $A = 0$ the condition defining $\mathbb{H}_0$ only depends on (θ, v), namely v must belong to the (open) ball

$$\mathscr{B}(\theta) := \{v \in \mathbb{R}^2 \, : \, |2v + \theta i|^2 < M^2 - (1 - \theta^2)\},$$

i.e. $\mathbb{H}_0(\theta, v) = \mathbb{R}^2$ (or $\emptyset$) if v belongs (or not) to $\mathscr{B}(\theta)$. On the other hand, for $A \neq 0$, $\mathbb{H}_A(\theta, v)$ is an (open) half-plane (except $\mathbb{H}_A(\theta, 0) = \mathbb{R}^2$).

In order to help better understand the set $U_{A,M}$ we provide several pictures (Fig. 5.2-5.4) of the slices $U_{A,M}(\theta, v)$, for some fixed A, M, θ, and different v's moving parallel to the real and imaginary axis. By symmetry ($U_{A,M}(\theta, -v^*) = -U_{A,M}(\theta, v)^*$) it is enough to consider $\Re v \geq 0$. We differentiate three cases: 1) $A = 0$, 2) $0 < |A < 1$ coupled with either 2.1) $M > M_*(A)$ or 2.2) $M < M_*(A)$ (cf. (5.63)).

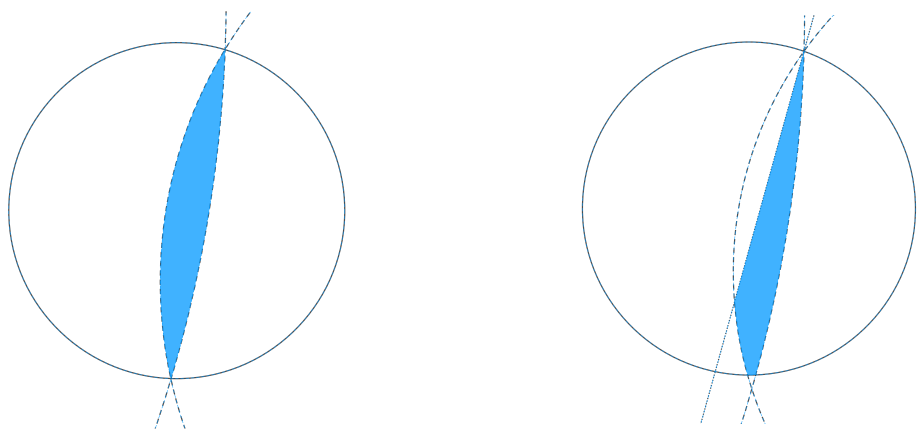


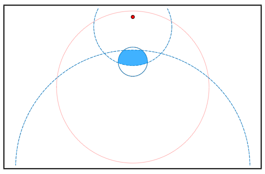
Figure 5.1: GeoGebra plot of the region $U_{A,M}(\theta, v)$ (blue) for some $(\theta, v) \in (-1, 1) \times \mathbb{R}^2$, $M > 1$, $A = 0$ (left) and $0 < |A| < 1$ (right), where we have added the circles $\partial\mathbb{B}(\theta, v)$ (solid), $\partial\mathbb{B}_-(\theta, v)$, $\partial\mathbb{B}_+(\theta, v)$ (dashed) and, for $A \neq 0$ (right), the line $\partial\mathbb{H}(\theta, v)$ (dotted).

1) Let $A = 0$. In this case, the region $U_{0,M}(\theta, v)$ does not collapse as v tends to $\partial\mathscr{B}(\theta)$ (cf. Fig. 5.2). In fact, $U_{0,M}(\theta, v)$ collapses if and only if $|\theta| \uparrow 1$ (i.e. u tends to K). In particular, as noted in [126], $\partial U_{0,M} \setminus K$ is locally the graph of a Lipschitz function.

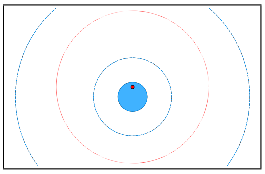
Since the case $A = 0$ is proved in [126], from now on we focus on $0 < |A| < 1$.

2) Let $0 < |A| < 1$. On the one hand, the half-plane $\mathbb{H}(\theta, v)$ causes that $U_{A,M}(\theta, v)$ collapses as $|v|$ grows, in contrast to the case $A = 0$ (cf. the last column of Fig. 5.2 and 5.3). On the other hand, we have to deal with the pinch singularity $Av + i = 0$. Given $\gamma > 0$ let us denote $S_\gamma := \{u \in \bar{U}_A \, : \, |Av + i| \leq \gamma\}$. The set S_0 ($\gamma = 0$) satisfies the following property. Let $(\theta, v, m) \in S_0$ with $|\theta| < 1$, i.e. $Av + i = 0$ and so $m = \theta v$. Then, it is straightforward to check that, for any $\square = \mathbb{H}, \mathbb{B}_-, \mathbb{B}_+$:

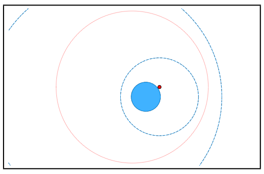
$$m \in \partial\square(\theta, v) \quad \Leftrightarrow \quad M = M_*(A). \tag{5.64}$$



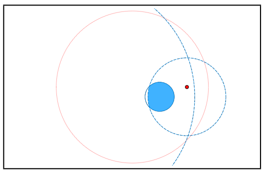
$v = -\frac{1}{2}\theta i + 1.8i$



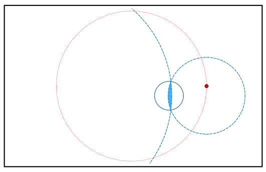
$v = -\frac{1}{2}\theta i$



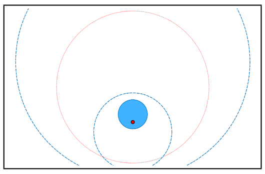
$v = -\frac{1}{2}\theta i + 0.7$



$v = -\frac{1}{2}\theta i + 1.4$



$v = -\frac{1}{2}\theta i + 1.953$



$v = -\frac{1}{2}\theta i - 0.9i$

Figure 5.2: Plots of $U_{0,M}(\theta, v)$ (cf. Fig. 5.1-left) for $A = 0$, $M = 4$, $\theta = \frac{1}{2}$ and different v's (red point) inside the circle $\partial\mathscr{B}(\theta)$ (red dotted).

Thus, for the particular value $M = M_*(A)$, the pinch singularity S_0 of U_A lies in the boundary of all the other new inequalities (5.61b)-(5.61d) defining $U_{A,M}$. For simplicity we omit this case.

2.1) Let $M > M_*(A)$. Then $U_{A,M}(\theta, v) = \mathbb{B}(\theta, v)$ in a neighborhood of $v = -\frac{1}{A}i$ (cf. Fig. 5.3). Therefore, there is $\gamma(A, M) > 0$ so that $S_\gamma \cap U_{A,M} = S_\gamma \cap U_A$ and thus the Λ-directions from Lemma 5.4.3 work in this region.

2.2) Let $M < M_*(A)$. Then $U_{A,M}(\theta, v) = \emptyset$ in a neighbourhood of $v = -\frac{1}{A}i$ (cf. Fig. 5.4). Therefore, there is $\gamma(A, M) > 0$ so that $S_\gamma \cap U_{A,M} = \emptyset$.

By 2.1) and 2.2), from now on we may assume that $|Av + i| > \gamma$ for some fixed $\gamma(A, M) > 0$. We remark in passing that, although we have removed the pinch singularity, it is not clear if $\partial U_{A,M} \setminus (K \cup S_\gamma)$ is locally the graph of a Lipschitz function (due to the collapse when $|v|$ grows) thus preventing from following the argument in [126].

Case $|Av + i| > \gamma$: From now on we focus on states $u = (\theta, v, m) \in U_M$ with $|Av + i| > \gamma$. In such case, there are $\omega \in \mathbf{D}$ and $\sigma_-, \sigma_+ \in \mathbb{D}$ so that m can be written as

$$\begin{aligned} m &= \theta v + (1 - \theta^2)\frac{Av + i}{1 + \omega\theta A}\omega \\ &= \mp v + \frac{1}{2}(1 \pm \theta)(M_\pm\sigma_\pm - i). \end{aligned} \tag{5.65}$$

Thus, ω, σ_-, σ_+ are related via

$$\pm v + (1 \mp \theta)\frac{Av + i}{1 + \omega\theta A}\omega = \frac{1}{2}(M_\pm\sigma_\pm - i). \tag{5.66}$$

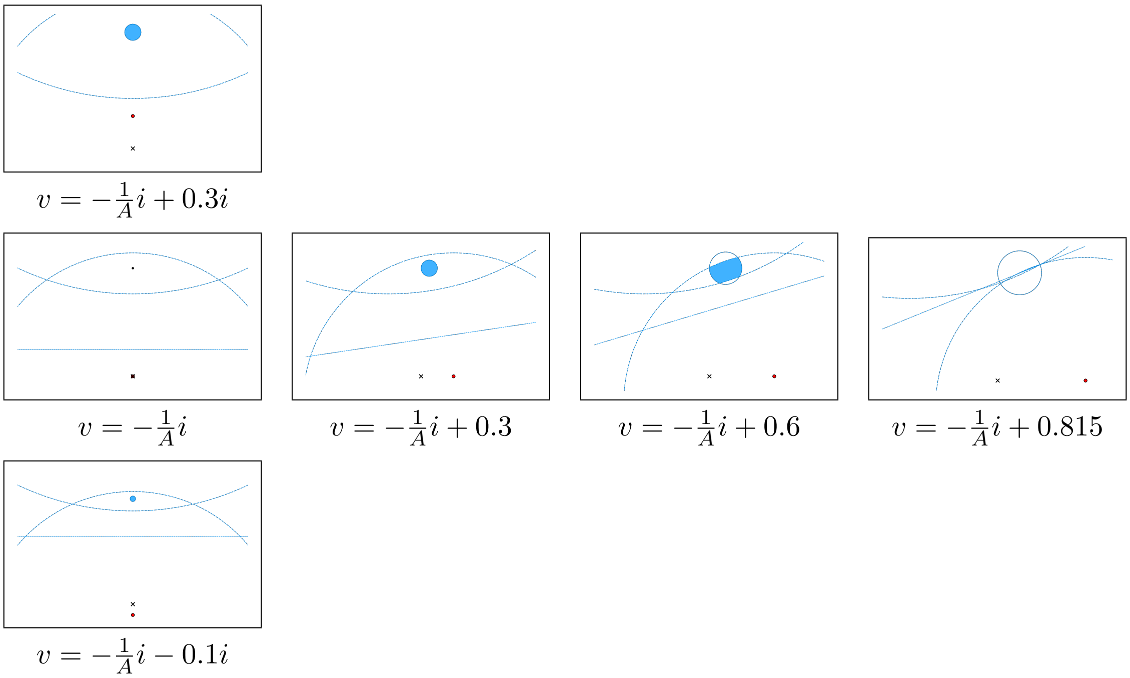


Figure 5.3: Plots of $U_{A,M}(\theta, v)$ (cf. Fig. 5.1-right) for $A = \frac{1}{2}$, $M = 4 > M_*(A)$, $\theta = \frac{1}{2}$ and different v's (red point) near the pinch singularity $Av + i = 0$ (cross) and far from it where $U_{A,M}(\theta, v)$ collapses.

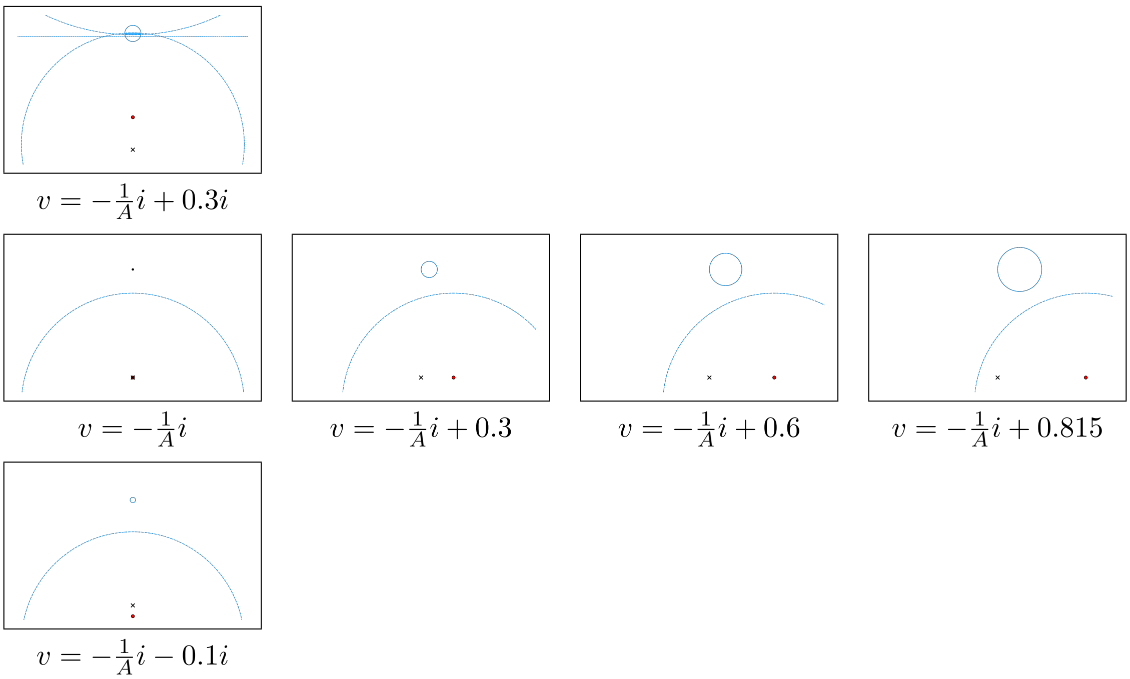


Figure 5.4: Plots of $U_{A,M}(\theta, v)$ (cf. Fig. 5.3) for $M = 3 < M_*(A)$.

By (5.65), we deduce that the identity (5.60) is equivalent to

$$(1-\theta^2)\left|\frac{Av+i}{1+\omega\theta A}\right|^2(1-|T\omega|^2)+(M^2-1-B(u)) \tag{5.67a}$$

$$=\frac{1-\theta}{2}(M^2-A)(1-|\sigma_-|^2)+\frac{1+\theta}{2}(M^2+A)(1-|\sigma_+|^2). \tag{5.67b}$$

In fact, (5.67) holds for all $u=(\theta,v,m)\in\bar{U}\setminus K$, with $\omega\in\bar{\mathbf{D}}$, $\sigma_-,\sigma_+\in\mathbb{R}^2$ defined via (5.65).

Since U_M is open, for every $u\in U_M$ and $\underline{u}\in\Lambda$ there is $\epsilon(u,\underline{u},U_M)>0$ so that $u_\lambda\equiv u+\lambda\underline{u}\in U_M$ for all $|\lambda|\leq\epsilon$. However, as in Lemma 5.4.2, we must choose $\underline{u}$ carefully in such a way that $\epsilon(1-\theta^2,A,M)$. Let us denote $\omega_\lambda\in\mathbf{D}$ and $\sigma_{\pm,\lambda}\in\mathbb{D}$ by the corresponding points that determine m_λ in the balls $\mathbb{B}(\theta_\lambda,v_\lambda)$ and $\mathbb{B}_\pm(\theta_\lambda,v_\lambda)$ respectively via (5.65).

Step 1. A change of variables: Let $\underline{u}(u)=(1,\underline{v},\underline{m})$ be the Λ-direction we want to construct. Thus, $\underline{v}=\underline{\omega}(A\underline{m}+i)$ with $(\underline{m},\underline{\omega})\in\mathbb{R}^2\times\mathbf{S}$ the degrees of freedom. Without loss of generality we take $\underline{m}=L^{-1}_{\theta\underline{\omega}}(\mathbf{v})$ in terms of $\mathbf{v}\in\mathbb{R}^2$. Inspired by Lemma 5.4.3, it is convenient to express w.l.o.g. this $\mathbf{v}$ as

$$\mathbf{v}(u,\underline{\mathbf{n}},\underline{\omega}):=v+\underline{\mathbf{n}}\frac{Av+i}{1+\omega\theta A}(\underline{\omega}-\omega), \tag{5.68}$$

in terms of some $\underline{\mathbf{n}}\in\mathbb{R}^2$ to be determined. Thus, if we denote (recall (5.44))

$$\mathbf{p}(u,\underline{\mathbf{n}},\underline{\omega}):=A\underline{m}+i=\frac{A\mathbf{v}+i}{1+\underline{\omega}\theta A}=\frac{Av+i}{1+\underline{\omega}\theta A}\left(1+\frac{A\underline{\mathbf{n}}(\underline{\omega}-\omega)}{1+\omega\theta A}\right), \tag{5.69}$$

the Λ-direction $\underline{u}$ is written as

$$\underline{v}=\underline{\omega}\mathbf{p},\qquad \underline{m}=\mathbf{v}-\theta\underline{v}, \tag{5.70}$$

in terms of $(\underline{\mathbf{n}},\underline{\omega})\in\mathbb{R}^2\times\mathbf{S}$, which shall be determined in the *step 2* and *3* respectively.

Step 2. Choice of $\underline{\mathbf{n}}$: Let us expand the condition $m_\lambda\in\mathbb{B}_\pm(\theta_\lambda,v_\lambda)$ in terms of λ:

$$\begin{aligned}2(m_\lambda\pm v_\lambda)+(1\pm\theta_\lambda)i&=2(m\pm v)+(1\pm\theta)i+\lambda(2(\underline{m}\pm\underline{v})\pm i)\\&=M_\pm(1\pm\theta_\lambda)\sigma_\pm+\lambda v_\pm(u,\underline{u}),\end{aligned} \tag{5.71}$$

where we have abbreviated (recall (5.65)-(5.70))

$$\begin{aligned}\frac{1}{2}v_\pm(u,\underline{u})&:=(\underline{m}\pm\underline{v})\mp\frac{1}{2}(M_\pm\sigma_\pm-i)\\&=(\mathbf{v}-v)\pm(1\mp\theta)\left(\frac{A\mathbf{v}+i}{1+\underline{\omega}\theta A}\underline{\omega}-\frac{Av+i}{1+\omega\theta A}\omega\right)\\&=\frac{1\pm\underline{\omega}A}{1+\underline{\omega}\theta A}(\mathbf{v}-v)\pm\frac{(1\mp\theta)(Av+i)}{(1+\underline{\omega}\theta A)(1+\omega\theta A)}(\underline{\omega}-\omega)\\&=\frac{(1\pm\underline{\omega}A)(Av+i)}{(1+\underline{\omega}\theta A)(1+\omega\theta A)}\left(\underline{\mathbf{n}}\pm\frac{1\mp\theta}{1\pm\underline{\omega}A}\right)(\underline{\omega}-\omega).\end{aligned} \tag{5.72}$$

From (5.71) we deduce that

$$(1-|\sigma_{\pm,\lambda}|^2) = (1-|\sigma_\pm|^2) - \tilde{\lambda}_\pm v_\pm \cdot (2\sigma_\pm + \tilde{\lambda}_\pm v_\pm), \tag{5.73}$$

with

$$\tilde{\lambda}_\pm \equiv \frac{\lambda}{M_\pm(1\pm\theta_\lambda)}.$$

Notice that $(1\pm\theta_\lambda) \geq \frac{1}{2}(1\pm\theta) \geq \frac{1}{2}(1-|\theta|)$ provided $|\lambda| \leq \frac{1}{2}(1-|\theta|)$.
The identities (5.72)(5.73) determines a good choice of $\underline{\mathbf{n}}$. More precisely, let us assume w.l.o.g. that $|\sigma_-| \leq |\sigma_+|$ (the case $|\sigma_+| < |\sigma_-|$ is totally analogous). Then, it is convenient to take (in fact necessary on $(\partial\mathbb{B}_+ \setminus \partial\mathbb{B})(\theta, v)$)

$$\underline{\mathbf{n}}(u,\underline{\omega}) = -\frac{1-\theta}{1+\underline{\omega}A}, \tag{5.74}$$

with $\underline{\omega}$ to be determined yet. With this choice of $\underline{\mathbf{n}}$, (5.72) reads as

$$v_+(u,\underline{\omega}) = 0, \qquad v_-(u,\underline{\omega}) = -\frac{4}{1+\underline{\omega}A}\frac{Av+i}{1+\omega\theta A}(\underline{\omega}-\omega), \tag{5.75}$$

and (5.69) reads as

$$\mathbf{p}(u,\underline{\omega}) = \frac{1+\omega A}{1+\underline{\omega}A}\frac{Av+i}{1+\omega\theta A} =: \frac{\mathbf{q}(u)}{1+\underline{\omega}A}, \tag{5.76}$$

where we have introduced $\mathbf{q}(u)$ as the part of $\mathbf{p}(u,\underline{\omega})$ independent of $\underline{\omega}$. Hence, by (5.75), (5.73) reads as $|\sigma_{+,\lambda}| = |\sigma_+|$, and so $m_\lambda \in \mathbb{B}_+(\theta_\lambda, v_\lambda)$ trivially for all $|\lambda| < (1-|\theta|)$.

In summary, we have seen that we can take $\underline{\mathbf{n}}$ (depending on whether $|\sigma_-| \leq |\sigma_+|$ or $|\sigma_+| < |\sigma_-|$[3]) in such a way that the condition $m_\lambda \in \mathbb{B}_+(\theta_\lambda, v_\lambda)$ (or $\mathbb{B}_-(\theta_\lambda, v_\lambda)$) holds for all $|\lambda| < (1-|\theta|)$. Thus, it remains to control the other three inequalities in (5.61), i.e. $\mathbb{B}_-$, $\mathbb{B}$ and $\mathbb{H}$.

Step 3. Choice of $\underline{\omega}$: By (5.73)(5.75), the condition $m_\lambda \in \mathbb{B}_-(\theta_\lambda, v_\lambda)$ can be written as

$$\tilde{\lambda}_- O(|T\underline{\omega} - T\omega|) < (1-|\sigma_-|^2). \tag{5.77}$$

Notice that, since $|Av+i| > \gamma$ and $|\theta_-| \leq |\theta_+|$, the identity (5.67) yields

$$\begin{aligned} \frac{1}{4}(1-\theta^2)\gamma^2(1-|T\omega|^2) &\leq (1-\theta^2)\left|\frac{Av+i}{1+\omega\theta A}\right|^2(1-|T\omega|^2) \\ &\leq (5.67\text{a}) = (5.67\text{b}) \\ &\leq (M^2+|A|)(1-|\sigma_-|^2). \end{aligned} \tag{5.78}$$

Since $\mathbf{v} = v + O(|T\underline{\omega} - T\omega|)$ (5.68), by elementary computations as in the *step 2* of the proof of Lemma 5.4.3, we deduce that the condition $m_\lambda \in \mathbb{B}(\theta_\lambda, v_\lambda)$ can be written as

$$\lambda O(|T\underline{\omega} - T\omega|) < (1-\theta^2)(1-|T\omega|^2). \tag{5.79}$$

[3] If $|\sigma_+| < |\sigma_-|$ we take $\underline{\mathbf{n}}(u,\underline{\omega}) = \frac{1+\theta}{1-\underline{\omega}A}$ and so (5.72) reads as $v_+(u,\underline{\omega}) = \frac{4}{1-\underline{\omega}A}\frac{Av+i}{1+\omega\theta A}(\underline{\omega}-\omega)$, $v_-(u,\underline{\omega}) = 0$ and (5.69) reads as $\mathbf{p}(u,\underline{\omega}) = \frac{1-\omega A}{1-\underline{\omega}A}\frac{Av+i}{1+\omega\theta A} =: \frac{\mathbf{q}(u)}{1-\underline{\omega}A}$ for a slightly different $\mathbf{q}$.

In summary, by (5.78), to guarantee that (5.77)(5.79) hold (for all $|\lambda|$ depending on $(1-\theta^2)$) it is enough to show that we can take $\underline{\omega}\in\mathbf{S}$ satisfying $|T\underline{\omega}-T\omega|\lesssim(1-|T\omega|)$ as $|T\omega|\uparrow 1$. This suggests to take $T\underline{\omega}$ by the projection $\frac{T\omega}{|T\omega|}$ as in Lemma 5.4.3. However, the last inequality (5.61b) restricts the set of admissible $\underline{\omega}$'s. Let us see it.

Let us expand the condition $m_\lambda\in\mathbb{H}(\theta_\lambda,v_\lambda)$ in terms of λ:

$$(M^2-1-B(u_\lambda))=(M^2-1-B(u))-\lambda b(u,\underline{\omega}), \tag{5.80}$$

where $b\equiv b_A$ is

$$\begin{aligned} b(u,\underline{\omega}) &:= 4\underline{v}\cdot(v+Am+\theta i+Ai)+4v\cdot(\underline{v}+A\underline{m}+i)\\ &= 4(\underline{\omega}\mathbf{p})\cdot\mathbf{b}+4v\cdot((\underline{\omega}+1)\mathbf{p})\\ &= 2\mathbf{p}\cdot(T\underline{\omega}^*(\mathbf{b}+v)-(\mathbf{b}-v)). \end{aligned} \tag{5.81}$$

Before continuing with the choice of $\underline{\omega}$, let us remark a difference to the case of equal viscosities. For $A=0$, the functions B_0, $\mathbf{b}_0$ and b_0 do not depend on m (equiv. ω). As a result, given $(\theta,v)\in(-1,1)\times\mathbb{R}^2$, the set of $\underline{\omega}$'s that can be used as $B_0(\theta,v)\uparrow M^2-1$ (i.e. v tends to $\partial\mathscr{B}(\theta)$) is more explicit, namely this is $\Omega_0(\theta,v)=\{\underline{\omega}\in\mathbf{S}\,:\,m_{\underline{\omega}}\equiv\theta v+(1-\theta^2)\underline{\omega}i\in(\underline{\mathbb{B}}_-\cap\underline{\mathbb{B}}_+)(\theta,v)\}$ (i.e. $m_{\underline{\omega}}\in(\partial\mathbb{B}\cap\underline{\mathbb{B}}_-\cap\underline{\mathbb{B}}_+)(\theta,v)$), independently of m. Thus, for each $m\in U_{0,M}(\theta,v)$, the choice of $\underline{\omega}$ in [126] is the minimizer of $|\underline{\omega}-\omega|$ in $\Omega_0(\theta,v)$. To conclude, Székelyhidi checked that the circles $\partial\mathbb{B}_\pm(\theta,v)$ intersect $\partial\mathbb{B}(\theta,v)$ transversally. For $A\neq 0$, the analogous set of $\underline{\omega}$'s depends on (θ,v,m), in terms of the proximity to the boundary of the half-plane $\mathbb{H}(\theta,v)$, and it is less explicit. In this regard, for $A\neq 0$, instead of figuring out how is $\Omega_A(\theta,v,m)$, we design a suitable $\underline{\omega}$ for each u separately.

As in [126], in order to choose $\underline{\omega}$ we distinguish three cases (see Fig. 5.5) depending on some parameter $0<\delta(1-\theta^2,A,M,\gamma)<M^2-1$ which shall be determined in the *step 4*.

1) If $M^2-1-B(u)>\delta$ (cf. Fig. 5.5-yellow) we can take directly $\underline{\omega}\in\mathbf{S}$ as in Lemma 5.4.3, that is $\underline{\omega}=0$ if $|T\omega|\leq\frac{1}{2}$ and $T\underline{\omega}=\frac{T\omega}{|T\omega|}$ if $\frac{1}{2}<|T\omega|<1$ (clearly $|T\underline{\omega}-T\omega|\lesssim(1-|T\omega|)$). Notice that there is $B(A,M)>0$ so that $|b(u,\underline{\omega})|\leq B$. Hence, by (5.80), $m_\lambda\in\mathbb{H}(\theta_\lambda,v_\lambda)$ for all $|\lambda|<\delta/B$.

2) Now let us suppose that $M^2-1-B(u)\leq\delta$.

2.1) In this case, if $(1-|T\omega|)>\delta$ (cf. Fig. 5.5-orange), then (5.77)(5.79) hold for all $|\lambda|\lesssim(1-\theta^2)^2\delta$. Thus, as we shall see in *step 4*, there exists $\underline{\omega}$ satisfying $b(u,\underline{\omega})=0$. With such choice, (5.80) reads as $B(u_\lambda)=B(u)$, and so $m_\lambda\in\mathbb{H}(\theta_\lambda,v_\lambda)$ trivially for all $|\lambda|<(1-|\theta|)$.

2.2) Finally let us suppose that $(1-|T\omega|)\leq\delta$ (cf. Fig. 5.5-red). As we have seen, on the one hand, if $m\in\partial\mathbb{H}(\theta,v)$ we have to take $\underline{\omega}$ satisfying $b(u,\underline{\omega})=0=:\alpha_{\mathbb{H}}(u)$. On the other hand, if $m\in\partial\mathbb{B}(\theta,v)$ we have to take $\underline{\omega}=\omega$. Furthermore, for any $m\in\partial\mathbb{B}(\theta,v)$ (not necessarily on $\bar{U}_M(\theta,v)$) by applying $\mathbf{v}(u,\omega)=v$, $v_\pm(u,\omega)=0$, Lemma 5.4.1c, (5.73) and (5.80), the coefficient of order 1 in λ of the identity (5.67) reads as

$$b(u,\omega)=\frac{1}{2}((M^2-A)(1-|\sigma_-|^2)-(M^2+A)(1-|\sigma_+|^2))=:\alpha_{\mathbb{B}}(u).$$

[5] Recall that $\varphi_{\theta A}\in\mathrm{Aut}(\mathbf{D})$ ($\subset$ Möbius transformations) and so it preserves circles.

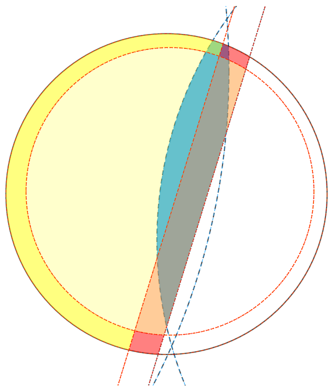


Figure 5.5: Plot of the various regions dividing $U_{A,M}(\theta,v)$ in terms of some $\delta>0$ small, for some $0<|A|<1$, $M>1$, $(\theta,v)\in(-1,1)\times\mathbb{R}^2$. Over $U_{A,M}(\theta,v)$ (cf. Fig. 5.1-right) we have overlapped: the circle[5] $(1-|T\omega|)=\delta$, the line $M^2-1-B(z)=\delta$, and the regions: 1) $M^2-1-B(z)>\delta$ (yellow: lighter if $(1-|T\omega|)>\delta$, darker if $(1-|T\omega|)\leq\delta$), 2) $M^2-1-B(z)\leq\delta$ coupled with either 2.1) $(1-|T\omega|)>\delta$ (orange) or 2.2) $(1-|T\omega|)\leq\delta$ (red).

Hence, both cases are compatible because, if $m\in(\partial\mathbb{B}\cap\partial\mathbb{H})(\theta,v)$, the identity (5.67) implies that $m\in(\partial\mathbb{B}_-\cap\partial\mathbb{B}_+)(\theta,v)$ too (cf. Fig. 5.1) and so $\alpha_{\mathbb{B}}(u)=0=\alpha_{\mathbb{H}}(u)$.
For states near the boundary, what we would like is to find $\underline{\omega}\in\mathbf{S}$ satisfying

$$b(u,\underline{\omega})=\alpha(u), \tag{5.82}$$

for some suitable interpolation $\alpha(u)$ from the values that b must take on the walls $\partial\mathbb{H}(\theta,v)$ and $\partial\mathbb{B}(\theta,v)$. In this regard, here we consider a convex combination of $\alpha_{\mathbb{B}}$ and $\alpha_{\mathbb{H}}$

$$\begin{aligned}\alpha(u)&:=\frac{(M^2-1-B(u))+d(u)}{(5.67\text{a})+d(u)}\alpha_{\mathbb{B}}(u)+\frac{(1-\theta^2)\left|\frac{Av+i}{1+\omega\theta A}\right|^2(1-|T\omega|^2)+d(u)}{(5.67\text{a})+d(u)}\alpha_{\mathbb{H}}(u)\\&=\frac{(M^2-1-B(u))+d(u)}{(5.67\text{a})+d(u)}\frac{1}{2}((M^2-A)(1-|\sigma_-|^2)-(M^2+A)(1-|\sigma_+|^2)),\end{aligned} \tag{5.83}$$

where we have introduced $d(u):=8\max\{1,|Av|\}\mathrm{dist}(m;U_M(\theta,v))$ to extend α on $\underline{\mathbb{B}}(\theta,v)\setminus U_M(\theta,v)$ (notice that $d(u)\geq 2|M^2-1-B(u)|$ on $\underline{\mathbb{B}}(\theta,v)\setminus U_M(\theta,v)$). For instance, if $m\in\partial\mathbb{B}_\pm(\theta,v)$ we have

$$\pm\alpha(u)=\frac{M^2-1-B(u)}{1\mp\theta}=\frac{1}{2}(M^2\mp A)(1-|\sigma_\mp|^2)-(1\pm\theta)\left|\frac{Av+i}{1+\omega\theta A}\right|^2(1-|T\omega|^2).$$

Hence, if there is such $\underline{\omega}\in\mathbf{S}$ satisfying (5.82) for (5.83), then (5.80) reads as

$$M^2-1-B(u_\lambda)=\frac{M^2-1-B(u)}{(5.67\text{b})}\left(\frac{1-\theta_\lambda}{2}(M^2-A)(1-|\sigma_-|^2)+\frac{1+\theta_\lambda}{2}(M^2+A)(1-|\sigma_+|^2)\right),$$

and so $m_\lambda\in\mathbb{H}(\theta_\lambda,v_\lambda)$ for all $|\lambda|<(1-|\theta|)$. Thus, it remains to show that there is $\underline{\omega}\in\mathbf{S}$ satisfying (5.82) and that the corresponding map $\omega\mapsto\underline{\omega}$ is Lipschitz (see (5.85)(5.86)).

Step 4. Lipschitz solution to $b(u,\underline{\omega}) = \alpha$: Firstly, let us determine the solvability of $b(z,\underline{\omega}) = \alpha$ for states $m \in \bar{\mathbb{B}}(\theta, v)$ and $\alpha \in \mathbb{R}$. By (5.76)(5.81), there is such $\underline{\omega} \in \mathbb{R}^2$ if and only if

$$\frac{T\underline{\omega}^*(\mathbf{b}+v) - (\mathbf{b}-v)}{1 + A\underline{\omega}^*} = \frac{1}{2}\frac{\alpha + \beta i}{\mathbf{q}^*},$$

or equivalently

$$(4\mathbf{q}^*(\mathbf{b}+v) - A(\alpha+\beta i))T\underline{\omega}^* = 4\mathbf{q}^*(\mathbf{b}-v) + (2-A)(\alpha+\beta i),$$

for some real β. Since we require $\underline{\omega} \in \mathbf{S}$, necessarily

$$|4\mathbf{q}^*(\mathbf{b}+v) - A(\alpha+\beta i)| = |4\mathbf{q}^*(\mathbf{b}-v) + (2-A)(\alpha+\beta i)|,$$

which turns out to be a quadratic equation for β, $a_2\beta^2 + a_1\beta + a_0 = 0$, where

$$\begin{aligned}
a_2 &= (1-A) > 0,\\
a_1 &= 4((1-A)\mathbf{b} - v)\cdot \mathbf{q}^\perp,\\
a_0 &= (1-A)\alpha^2 + 4((1-A)\mathbf{b} - v)\cdot \mathbf{q}^\perp\alpha - 4B(u)|\mathbf{q}|^2.
\end{aligned}$$

The discriminant of this quadratic equation verifies

$$\Delta(u,\alpha) = a_1^2 - 4a_2a_0 \geq 16(1-A)B(u)|\mathbf{q}(u)|^2 + O(\alpha).$$

In particular, if $B(u) \geq M^2 - 1 - \delta > 0$, for $\alpha = 0$ we have $\Delta(u,0) > 0$ and so there exists $\underline{\omega} \in \mathbf{S}$ satisfying $b(u,\underline{\omega}) = 0$. Now let $\alpha(u)$ given in (5.83). Notice that this can be bounded by

$$|\alpha(u)| \leq \frac{1}{2}(M^2 + |A|)(|1 - |\sigma_-|^2| + |1 - |\sigma_+|^2|).$$

Hence, since $|\mathbf{q}(u)| \geq \frac{1-|A|}{1+|A|}\gamma$, there is a constant $C(A, M, \gamma) > 0$ so that

$$\Delta(u,\alpha(u)) \geq 4(1-A)(M^2-1)\left(\frac{1-|A|}{1+|A|}\gamma\right)^2 > 0,$$

for all $m \in \underline{\mathbb{B}}(\theta, v)$ in the intersection of the half-plane $B(u) \geq \frac{1}{2}(M^2-1)$ and the annuli $|1 - |\sigma_-|^2|, |1 - |\sigma_+|^2| \leq C$. Therefore, in this region $L \equiv L_{A,M,\gamma}$

$$L(\theta, v) := \{m \in \underline{\mathbb{B}}(\theta, v) \,:\, B(u) \geq \tfrac{1}{2}(M^2-1),\, |1 - |\sigma_-|^2|, |1 - |\sigma_+|^2| \leq C\}$$

there are two ($s \in \{-1, 1\}$) solutions $T\underline{\omega}_s = q_s(u)$ to $b(u,\underline{\omega}) = \alpha(u)$ given by

$$q_s(u) := \frac{4\mathbf{q}(u)(\mathbf{b}(u) - v)^* + (2-A)(\alpha - \beta_s i)(u)}{4\mathbf{q}(u)(\mathbf{b}(u) + v)^* - A(\alpha - \beta_s i)(u)}, \tag{5.84}$$

where

$$\beta_s(u) := \frac{-a_1(u) + s\sqrt{\Delta(u,\alpha(u))}}{2a_2}.$$

Furthermore, since $\Delta(u,\alpha(u)) \gg 0$, the square root of Δ gives no problem and so the map $T\omega \mapsto q_s(\theta, v; T\omega)$ is Lipschitz in this region. In particular, we select the sign $s \in \{-1, 1\}$ that minimizes $|T\underline{\omega}_s - T\omega|$.

Finally, let $m \in U_M(\theta, v)$ with $|\sigma_-| \leq |\sigma_+|$ and $M^2 - 1 - B(u), 1 - |T\omega| \leq \delta$. Notice that the identity (5.67) yields

$$\frac{1 \pm \theta}{2}(M^2 \pm A)(1 - |\sigma_\pm|^2) \leq (5.67\text{b}) = (5.67\text{a}) = O(\delta).$$

Hence, we can take $\delta = D(1-\theta^2)(M^2-1)$ for some constant $0 < D(A, M, \gamma) < \frac{1}{2}$ in such a way that $m \in L(\theta, v)$. In addition, we can take D so that the projection m_0 of m into $\partial\mathbb{B}(\theta, v)$ given by $T\omega_0 = \frac{T\omega}{|T\omega|}$ also satisfies $m_0 \in L(\theta, v)$. Recall that, by construction, $b(u_0, \omega_0) = \alpha(u_0)$ since $m_0 \in \partial\mathbb{B}(\theta, v)$. Thus, for some $s(u) \in \{-1, 1\}$,

$$|T\underline{\omega} - T\omega_0| = |q_s(u) - q_s(u_0)| \lesssim |T\omega - T\omega_0|, \tag{5.85}$$

and so

$$|T\underline{\omega} - T\omega| \leq |T\underline{\omega} - T\omega_0| + |T\omega - T\omega_0| \lesssim |T\omega - T\omega_0| = (1 - |T\omega|). \tag{5.86}$$

If $|\sigma_+| < |\sigma_-|$ the formulas in *step 4* are slightly different but the argument does not change. This concludes the proof. □

5.4.3 The Λ-lamination hull

In this section we prove that $K^{lc,\Lambda} = \bar{U}$ and $(K_M)^{lc,\Lambda} = \bar{U}_M$.

Lemma 5.4.8. *Let $u_0 \in K$ and $u_1 \in K^{1,\Lambda}$ satisfying $u_1 - u_0 \in \Lambda$. Then, the segment $[u_0, u_1] = \{u_0 + \tau(u_1 - u_0) : \tau \in [0,1]\}$ lies in $\bar{U}$.*

Proof. Recall that, by Lemma 5.4.1e: $u_0, u_1 \in K$ s.t. $u_1 - u_0 \in \Lambda \Rightarrow [u_0, u_1] \subset \partial U$.
Now, let $u_0 = (\theta_0, v_0, m_0) \in K$ and $u_1 = (\theta_1, v_1, m_1) \in K^{1,\Lambda} \setminus K$, that is, $|\theta_0| = 1$, $|\theta_1| < 1$ and

$$m_0 = \theta_0 v_0, \qquad m_1 = \theta_1 v_1 + \frac{(1 - (\theta_1)^2)(Av_1 + i)}{1 + \underline{\omega}_1 \theta_1 A}\underline{\omega}_1, \tag{5.87}$$

for some $\underline{\omega}_1 \in \mathbf{S}$. Let us suppose that $\underline{u} \equiv u_1 - u_0 \in \Lambda$, that is, $\underline{v} = \underline{\omega}(A\underline{m} + \underline{\theta} i)$ for some $\underline{\omega} \in \mathbf{S}$. We want to show that the intermediate states $u_\tau \equiv u_0 + \tau\underline{u}$ belong to $\bar{U}$ for all $\tau \in (0,1)$. We split the proof in two steps. Firstly (*step 1*) we prove the statement by assuming a claim. Secondly (*step 2*) this claim is proved by elementary computations.

Step 1. Claim: Given $\tau \in (0,1)$, there is $\omega_\tau \in \mathbb{R}^2$ satisfying

$$(1 + \omega_\tau \theta_\tau A)(m_\tau - \theta_\tau v_\tau) = (1 - (\theta_\tau)^2)(Av_\tau + i)\omega_\tau, \tag{5.88}$$

if and only if

$$(Av_1 + i)((\beta_\tau - \beta)\omega_\tau - (\beta_\tau \underline{\omega} - \beta\underline{\omega}_1)) = 0, \tag{5.89}$$

where we have abbreviated

$$\beta_\tau \equiv (\theta_1 - \theta_0)\alpha_1(1-\tau), \qquad \alpha_1 \equiv 1 - \underline{\omega}_1 \theta_0 A, \tag{5.90a}$$

$$\beta \equiv (\theta_1 + \theta_0)\alpha, \qquad \alpha \equiv 1 - \underline{\omega}\theta_0 A. \tag{5.90b}$$

(Notice that $\alpha,\alpha_1,\beta,\beta_\tau\neq 0$). We shall prove this equivalence in the *step 2*.
Assume that this claim is true. Then, if $Av_1+i=0$, (5.89) holds trivially for every $\omega_\tau\in\mathbf{S}$ ($\Rightarrow u_\tau\in\partial U$ by Lemma 5.4.1e). Now let us assume that $Av_1+i\neq 0$. Hence, (5.89) holds if and only if

$$(\beta_\tau-\beta)\omega_\tau=\beta_\tau\underline{\omega}-\beta\underline{\omega}_1,$$

or equivalently (by applying the translation operator T (5.33))

$$(\beta_\tau-\beta)T\omega_\tau=\beta_\tau T\underline{\omega}-\beta T\underline{\omega}_1. \tag{5.91}$$

A priori there could be some (unique) $\tau\in(0,1)$ satisfying $\beta_\tau=\beta$. However, since $\bar{U}$ is closed and $\tau\mapsto u_\tau$ is continuous, it is enough to prove the statement for the remainder τ's satisfying $\beta_\tau\neq\beta$. For those τ's, (5.91) determines ω_τ:

$$T\omega_\tau=\frac{\beta_\tau T\underline{\omega}-\beta T\underline{\omega}_1}{\beta_\tau-\beta}.$$

Hence, since $|T\underline{\omega}|=|T\underline{\omega}_1|=1$, we have (recall (5.90))

$$\begin{aligned}|T\omega_\tau|^2&=1+2\frac{\beta_\tau\cdot\beta-(\beta_\tau T\underline{\omega})\cdot(\beta T\underline{\omega}_1)}{|\beta_\tau-\beta|^2}\\&=1-2(1-\tau)(1-(\theta_1)^2)\frac{\alpha_1\cdot\alpha-(\alpha_1T\underline{\omega})\cdot(\alpha T\underline{\omega}_1)}{|\beta_\tau-\beta|^2}.\end{aligned} \tag{5.92}$$

Finally, by applying

$$\begin{aligned}4\alpha_1\alpha^*&=(2+(1-T\underline{\omega}_1)\theta_0A)(2+(1-T\underline{\omega}^*)\theta_0A)\\&=(2+\theta_0A)^2+(\theta_0A)^2T\underline{\omega}_1T\underline{\omega}^*-\theta_0A(2+\theta_0A)(T\underline{\omega}_1+T\underline{\omega}^*),\end{aligned}$$

we get

$$\begin{aligned}\alpha_1\cdot\alpha-(\alpha_1T\underline{\omega})\cdot(\alpha T\underline{\omega}_1)&=\Re((\alpha_1\alpha^*)(1-T\underline{\omega}T\underline{\omega}_1^*))\\&=\underbrace{\frac{1}{4}((2+\theta_0A)^2-(\theta_0A)^2)}_{=1+\theta_0A}(1-T\underline{\omega}\cdot T\underline{\omega}_1)\geq 0.\end{aligned}$$

Therefore, (5.92) yields $|T\omega_\tau|\leq 1$ ($\Rightarrow u_\tau\in\bar{U}$ by Lemmas 5.4.1d and 5.4.2d).

Step 2. Proof of the claim: On the one hand, $\underline{\theta}=\theta_1-\theta_0$, $\underline{v}=v_1-v_0$ and, by (5.87),

$$\underline{m}=m_1-m_0=\theta_0\underline{v}+\underline{\theta}\left(v_1-\frac{(\theta_1+\theta_0)(Av_1+i)}{1+\underline{\omega}_1\theta_1A}\underline{\omega}_1\right). \tag{5.93}$$

On the other hand, by applying (5.93) into the condition $\underline{v}=\underline{\omega}(A\underline{m}+\underline{\theta}i)$ we get

$$(\underbrace{1-\underline{\omega}\theta_0A}_{=\alpha})\underline{v}=\underline{\omega}\underline{\theta}\frac{(\overbrace{1-\underline{\omega}_1\theta_0A}^{=\alpha_1})(Av_1+i)}{1+\underline{\omega}_1\theta_1A}. \tag{5.94}$$

Let us abbreviate $\langle u\rangle\equiv u_1+u_0$ and

$$\mathbf{f}\equiv\frac{Av_1+i}{\alpha(1+\underline{\omega}_1\theta_1A)}.$$

(Notice that: $\mathbf{f} = 0 \Leftrightarrow Av_1 + i = 0$). Thus, (5.93)(5.94) read as

$$\underline{v} = \underline{\theta}\alpha_1\underline{\omega}\mathbf{f}, \qquad \underline{m} = \theta_0\underline{v} + \underline{\theta}(v_1 - \langle\theta\rangle\alpha\underline{\omega}_1\mathbf{f}).$$

Let us expand the factors of (5.88) in terms of τ. They are

$$m_\tau - \theta_\tau v_\tau = \underbrace{m_0 - \theta_0 v_0}_{=0} + \tau(\underbrace{\underline{m} - \theta_0\underline{v} - \underline{\theta}v_0}_{=\underline{\theta}(\underline{v} - \langle\theta\rangle\alpha\underline{\omega}_1\mathbf{f})}) - \tau^2\underline{\theta}\underline{v} = \tau\underline{\theta}(\underbrace{(1-\tau)\underline{\theta}\alpha_1}_{=\beta_\tau}\underline{\omega} - \underbrace{\langle\theta\rangle\alpha}_{=\beta}\underline{\omega}_1)\mathbf{f},$$

$$1 - (\theta_\tau)^2 = (\underbrace{\theta_0 - \theta_\tau}_{=-\tau\underline{\theta}})(\theta_0 + \theta_\tau),$$

and

$$\begin{aligned} Av_\tau + i &= (Av_1 + i) - A(1-\tau)\underline{v} \\ &= (\alpha(1 + \underline{\omega}_1\theta_1 A) - A(1-\tau)\underline{\theta}\alpha_1\underline{\omega})\mathbf{f} \\ &= (\alpha\alpha_1 - A(\underbrace{(1-\tau)\underline{\theta}\alpha_1}_{=\beta_\tau}\underline{\omega} - \underbrace{\langle\theta\rangle\alpha}_{=\beta}\underline{\omega}_1))\mathbf{f}. \end{aligned}$$

Hence, the equation (5.88) reads as

$$(1 + \omega_\tau\theta_\tau A)\tau\underline{\theta}(\beta_\tau\underline{\omega} - \beta\underline{\omega}_1)\mathbf{f} = \tau\underline{\theta}(\theta_0 + \theta_\tau)(A(\beta_\tau\underline{\omega} - \beta\underline{\omega}_1) - \alpha\alpha_1)\mathbf{f}\omega_\tau,$$

or equivalently ($\tau\underline{\theta} \neq 0$)

$$(\beta_\tau\underline{\omega} - \beta\underline{\omega}_1)\mathbf{f} = (\theta_0 A(\beta_\tau\underline{\omega} - \beta\underline{\omega}_1) - (\theta_0 + \theta_\tau)\alpha\alpha_1)\omega_\tau\mathbf{f}. \tag{5.95}$$

Finally, by splitting $(\theta_0 + \theta_\tau) = \langle\theta\rangle - (1-\tau)\underline{\theta}$, we have

$$(\theta_0 + \theta_\tau)\alpha\alpha_1 = (1 - \underline{\omega}_1\theta_0 A)\underbrace{\langle\theta\rangle\alpha}_{=\beta} - (1 - \underline{\omega}\theta_0 A)\underbrace{(1-\tau)\underline{\theta}\alpha_1}_{=\beta_\tau},$$

and so (5.95) is equivalent to (5.89). □

Proposition 5.4.1. $K^{lc,\Lambda} = K^{2,\Lambda} = \bar{U}$.

Proof. Firstly (*step 1*) we prove that $K^{2,\Lambda} = \bar{U}$. Secondly (*step 2*) we deduce that $K^{lc,\Lambda} = K^{2,\Lambda}$.

Step 1. $K^{2,\Lambda} = \bar{U}$: Since U is open and $\partial U = K^{1,\Lambda}$ (Lemma 5.4.1), this is equivalent to prove that $K^{2,\Lambda} \setminus K^{1,\Lambda} = U$.
By definition (5.41) and Lemma 5.4.1g a state $u = (\theta, v, m) \in [-1,1] \times \mathbb{R}^2 \times \mathbb{R}^2$ belongs to $K^{2,\Lambda} \setminus K^{1,\Lambda}$ if and only if $g(u) \neq 0$ and there are $0 \neq \underline{u} \in \Lambda$ and $\lambda_- < 0 < \lambda_+$ satisfying

$$|\theta_{\lambda_\pm}| \leq 1, \qquad g(u_{\lambda_\pm}) = 0,$$

where $u_\lambda \equiv u + \lambda\underline{u}$. Since $\underline{u} \in \Lambda$, notice that

$$\det L(u_\lambda) = \text{quadratic} + \lambda^3 \underbrace{\det L(\underline{u})}_{=0}.$$

Then, by Lemma 5.4.1g, the polynomial $p : \lambda \mapsto g(u_\lambda)$ is cubic

$$p(u,\underline{u};\lambda) := g(u_\lambda) = \sum_{j=0}^{3} a_j(u,\underline{u})\lambda^j. \tag{5.96}$$

Step 1.1. $\bar{U} \subset K^{2,\Lambda}$: The analysis of (5.96) is easier for $\underline{u} \in \Lambda_0$ because p is quadratic ($a_3 = 0$) in such case. Moreover, since $\underline{\theta} = 0$ and $\underline{v} = -A\underline{m}$, the second coefficient is strictly positive

$$a_2 = (\underline{m} + A\underline{v} - \theta(\underline{v} + A\underline{m})) \cdot (\underline{m} - \theta\underline{v}) = (1 - A^2)(1 + \theta A)|\underline{m}|^2 > 0.$$

Hence, p has two real roots of different sign if and only if $g(u) = a_0 < 0$ ($u \in U$). Therefore, $\bar{U} = (K^{1,\Lambda})^{1,\Lambda_0} \subset K^{2,\Lambda}$ (As a curiosity observe that, since g is Λ_0-convex, $\bar{U} = (K^{1,\Lambda})^{\Lambda_0}$).

Step 1.2. $K^{2,\Lambda} \subset \bar{U}$. Since $K^{1,\Lambda} = K^{1,\Lambda_1}$ (Lemma 5.4.1), by the *step 1* we only need to check that $K^{2,\Lambda_1} \setminus K^{1,\Lambda_1} \subset U$.

Let $u = (\theta, v, m) \in K^{2,\Lambda_1} \setminus K^{1,\Lambda_1}$. By hypothesis, $g(u) \neq 0$ and there are $0 \neq \underline{u} \in \Lambda_1$ with $\underline{\theta} = 1$ and $\lambda_- < 0 < \lambda_+$ satisfying $|\theta + \lambda_\pm| \leq 1$ and $p(u,\underline{u};\lambda_\pm) = g(u_{\lambda_\pm}) = 0$. Notice that necessarily $|\theta| < 1$. If we abbreviate $u^\pm \equiv u_{(\pm 1 - \theta)} = u + (\pm 1 - \theta)\underline{u}$, then $\theta^\pm = \pm 1$ and Lemma 5.4.1g yields

$$p(u,\underline{u};\pm 1 - \theta) = g(u^\pm) = (1 \mp A)|m^\pm \mp v^\pm|^2 \geq 0.$$

If both $p(u,\underline{u};\pm 1 - \theta) > 0$ necessarily $g(u) = p(u,\underline{u};0) < 0$, otherwise we would deduce that $p'(u,\underline{u};\cdot)$ has at least 3 roots in $[-1-\theta, 1-\theta]$. If both $p(u,\underline{u};\pm 1 - \theta) = 0$ we would have $u^\pm = (\pm 1, v^\pm, \pm v^\pm) \in K$, and so $u \in K^{1,\Lambda_1}$. If only one of $p(u,\underline{u};\pm 1 - \theta)$ is zero, then u is a Λ-convex combination of a state in K and other in $K^{1,\Lambda_1} \setminus K$. Thus, by Lemma 5.4.8, $u \in \bar{U}$.

Step 2. $K^{lc,\Lambda} = K^{2,\Lambda}$: It is a general fact in Lamination Theory that, for any closed K, the following holds: $K^{1,\Lambda} \setminus K = (\partial K)^{1,\Lambda} \setminus K$. Hence, since $\partial(K^{2,\Lambda}) = \partial U = K^{1,\Lambda}$, we deduce that $K^{3,\Lambda} \setminus K^{2,\Lambda} = \emptyset$. Therefore, inductively $K^{n,\Lambda} = K^{2,\Lambda}$ for all $n \geq 3$. □

Proposition 5.4.2. *Let $M > 1$. Then $(K_M)^{lc,\Lambda} = \bar{U}_M$.*

Proof. Step 1. $(K_M)^{lc,\Lambda} \subset \bar{U}_M$: It follows from: $\bar{U}$ is Λ-lamination convex, (5.61b) defines the sublevel set of a Λ-convex (indeed Λ-affine) function, and (5.61c)-(5.61d) define sublevel sets of convex functions.

Step 2. $\bar{U}_M \subset (K_M)^{lc,\Lambda}$: As in [126], it follows from the Krein-Milman type theorem in the context of Λ-convexity [84, Lemma 4.16], because, as we saw in Lemma 5.4.7, for all $u \in \partial U_M \setminus K_M$ there is $0 \neq \underline{u} \in \Lambda$ such that $u \pm \underline{u} \in \bar{U}_M$ (i.e. u is not an extreme point of $\bar{U}_M$). More precisely, let $u = (\theta, v, m) \in \partial U_M \setminus K_M$. As in *step 1* in the proof of Lemma 5.4.7, we take $\underline{u}$ in terms of $(\underline{\mathbf{n}}, \underline{\omega}) \in \mathbb{R}^2 \times \mathbf{S}$ to be determined. If $m \in \partial\mathbb{B}$ ($\omega \in \mathbf{S}$) it is enough to take $\underline{\omega} = \omega$. Otherwise ($m \notin \partial\mathbb{B}$) we may assume w.l.o.g. that $m \notin \partial\mathbb{B}_-$. If $m \in \partial\mathbb{H}$ we take $\underline{\omega}$ satisfying $b(u,\underline{\omega}) = 0$ (5.80). If $m \in \partial\mathbb{B}_+$ we take $\underline{\mathbf{n}}$ as in (5.74). □

Remark 5.4.4. Notice that we are not excluding the case $M = M_*(A)$ in Proposition 5.4.2. Although we believe that in this case Lemma 5.4.7 holds too, we have chosen to exclude it in Lemma 5.4.7 for simplicity.

Remark 5.4.5. In [126] the identity $\bar{U}_0 = K^{\Lambda_0}$ (and also $\bar{U}_{0,M} = (K_{0,M})^{\Lambda_0}$) follows from the fact that f_0 is Λ_0-convex. However, f_A (Lemma 5.4.1f) is not Λ_A-convex for $A \neq 0$: Let $u_0 = (0, -i/A, 0) \in \bar{U}$ and $\underline{u}_0 = (1,0,0) \in \Lambda$ ($(\underline{m}, \underline{\omega}) = (0,0) \in \mathbb{R}^2 \times \mathbf{S}$). Then, the function

$$h_A(\lambda) := f_A(u_0 + \lambda\underline{u}_0) = 2|1 - \lambda A|\frac{|\lambda|}{|A|}, \tag{5.97}$$

is not convex since

$$\partial_\lambda^2 h_A(\lambda) = -4\mathrm{sgn}(\lambda A), \quad 0 < |\lambda| < 1/A.$$

Notice that this does not imply that $\bar{U}_A \subsetneq K^{\Lambda_A}$. In general, $\bar{U}_A$ can be expressed as $\{u \in [-1,1] \times \mathbb{R}^2 \times \mathbb{R}^2 \, : \, c_A(u) f_A(u) \leq 0\}$ for all $c_A > 0$ on $[-1,1] \times \mathbb{R}^2 \times \mathbb{R}^2$. Thus, to prove that $\bar{U}_A = K^{\Lambda_A}$ it is enough to find a correcting factor $c_A > 0$ making $c_A f_A$ Λ_A-convex on $[-1,1] \times \mathbb{R}^2 \times \mathbb{R}^2$. For instance, $c_A(u) = 1/(1 - \theta A)$ repairs the counterexample (5.97) since $(c_A f_A)(u_0 + \lambda \underline{u}) = 2|\lambda|/|A|$. However, it seems hard to check if $c_A f_A$ is Λ_A-convex. Still we conjecture that $\bar{U}_A$ is indeed K^{Λ_A} and also closed under weak*-convergence, thus representing the full relaxation of IPM_A in analogy with the case $A = 0$.

5.5 Toy random walk

In this section we introduce a toy random walk to illustrate how these Θ_A-mixing solutions may look like (see Fig. 5.6-5.11) and, at the same time, to give somehow an intuitive idea of the interplay between the unpredictable nature at the microscopic level of the mixing phenomenon and the deterministic point of view at the mesoscopic scale. This is also motivated by the relaxation approach of Otto [116, sec. 2].

Otto's approach

Roughly speaking, by passing from the Eulerian (phase $\theta(t,x)$) to the Lagrangian (flow map $\Phi(t,x)$) point of view, Otto rewrote the Muskat problem as a gradient flux for Φ w.r.t. the gravitational potential energy E with the following physical interpretation: "Given θ° (5.6), the phase distribution θ advected by the flow $(\theta(t, \Phi(t)) = \theta^\circ)$ aims at minimizing E by transforming it into kinetic energy, which then is dissipated by friction when forcing the fluid through the porous medium". A natural discretization in time intervals of size h yields a recurrence $\Phi_h^k \rightsquigarrow \Phi_h^{k+1}$ starting from $\Phi_h^0 = \mathrm{id}$ that leads an approximate time-discrete solution $\Phi_h = (\Phi_h^k)_k$, where Φ_h^{k+1} is the unique solution of a variational problem defined in terms of Φ_h^k. As he noted, Φ_h^1 is not one-to-one, thus preventing (a priori) from defining the corresponding θ_h^1 by advection. Nevertheless, by subdividing the space in a grid of size r, each Φ_h^k can be approximated by a (minimizing) sequence of permutations $\Phi_{h,r}^k$ of this partition. Then, each $\Phi_{h,r}^k$ defines a $\{-1,1\}$-valued discrete phase distribution $\theta_{h,r}^k = (\theta^\circ)_r \circ (\Phi_{h,r}^k)^{-1}$ where $(\theta^\circ)_r$ is a sampling of θ°. It is interesting to point that $\Phi_{h,r}^1$ (and so $\theta_{h,r}^1$) breaks the planar symmetry of (5.6) and consequently is not unique. Despite this lack of uniqueness, Otto showed that $\theta_{h,r}^k \overset{*}{\rightharpoonup} \theta_h^k = (\Phi_h^k)^\sharp(\theta^\circ) \equiv$ push-forward of θ° under Φ_h^k, which allows to interpret θ_h^k as the average in space of the actual phase distribution. At the same time, θ_h^k is the unique solution of a convex variational problem, linked with the one for Φ_h^k through Optimal Transport Theory. To conclude Otto proved that θ_h converges in $L_t^\infty L^1$ to the (unique) entropy solution Θ_A (5.14) of the conservation law (5.15).

Toy random walk

As in [116], we discretize in time intervals of size $h = \triangle t$ and we subdivide the domain in a grid of size $r = \triangle x_i$ whose center points form the lattice $r(\mathbb{Z}^2 + \frac{1}{2}i) = \{x_{s,j} \equiv r(s, j + \frac{1}{2}) \, : \, s, j \in \mathbb{Z}\}$.

Take a sample of θ° (5.6)

$$\theta^{(0)}(x_{s,j}) = \begin{cases} +1, & j > 0, \\ -1, & j < 0. \end{cases} \tag{5.98}$$

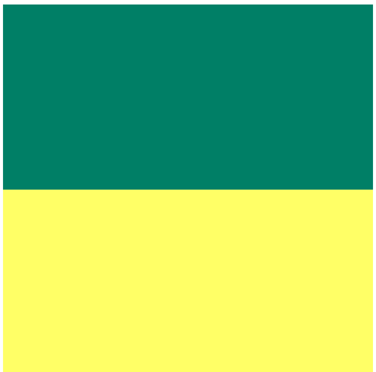


Figure 5.6: The unstable planar phase distribution.

Then, we interpret the conservation of mass and volume by setting that two close different “molecules” may interchange their positions if the heavier is above the lighter, i.e. if their state is unstable due to gravity. Darcy’s law is interpreted by setting that such interchange happens with some probability

$$p_j^{(k)} \equiv \text{probability of interchange between lines } j \leftrightarrow j-1 \text{ at time } k+1, \tag{5.99}$$

depending on the Atwood number A and in terms of the proximity to the rest molecules of the same fluid respectively. Note that, by simplicity, we are considering $p_j^{(k)}$ independent of s due to the planar symmetry of (5.98). This induces a time-discrete stochastic process $\{\theta_{s,j}^{(k)}\}_{k\geq 0}$ where $\theta_{s,j}^{(k)} \equiv \theta^{(k)}(x_{s,j})$ is the $\{-1,1\}$-valued random variable. In this way, (5.99) reads as

$$p_j^{(k)} = P(\text{interchange} \checkmark \,|\, \theta_{s,j}^{(k)} = 1,\, \theta_{s,j-1}^{(k)} = -1),$$

while the probability of interchange in the remaining situations is zero. We are interested in the deterministic value

$$\bar{\theta}_j^{(k)} := E(\theta_{s,j}^{(k)}) = d_{j,+}^{(k)} - d_{j,-}^{(k)}, \tag{5.100}$$

where

$$d_{j,\pm}^{(k)} := P(\theta_{s,j}^{(k)} = \pm 1).$$

This $d_j^{(k)}$ can be computed recursively

$$\begin{aligned} d_{j,+}^{(k+1)} &= \underbrace{d_{j,+}^{(k)} d_{j-1,+}^{(k)}}_{\text{stable}} + \underbrace{(1-p_j^{(k)}) d_{j,+}^{(k)} d_{j-1,-}^{(k)}}_{\substack{\text{unstable} \\ \text{interchange ✗}}} + \underbrace{p_{j+1}^{(k)} d_{j+1,+}^{(k)} d_{j,-}^{(k)}}_{\substack{\text{unstable} \\ \text{interchange ✓}}} \\ &= \underbrace{d_{j,+}^{(k)}}_{✗} + \underbrace{p_{j+1}^{(k)} d_{j+1,+}^{(k)} d_{j,-}^{(k)}}_{\substack{\text{interchange ✓} \\ +\text{ increases}}} - \underbrace{p_j^{(k)} d_{j,+}^{(k)} d_{j-1,-}^{(k)}}_{\substack{\text{interchange ✓} \\ +\text{ reduces}}}, \end{aligned}$$

and analogously

$$\begin{aligned} d_{j,-}^{(k+1)} &= \underbrace{d_{j+1,-}^{(k)} d_{j,-}^{(k)}}_{\text{stable}} + \underbrace{(1-p_{j+1}^{(k)}) d_{j+1,+}^{(k)} d_{j,-}^{(k)}}_{\substack{\text{unstable} \\ \text{interchange ✗}}} + \underbrace{p_j^{(k)} d_{j,+}^{(k)} d_{j-1,-}^{(k)}}_{\substack{\text{unstable} \\ \text{interchange ✓}}} \\ &= \underbrace{d_{j,-}^{(k)}}_{✗} - \underbrace{p_{j+1}^{(k)} d_{j+1,+}^{(k)} d_{j,-}^{(k)}}_{\substack{\text{interchange ✓} \\ -\text{ reduces}}} + \underbrace{p_j^{(k)} d_{j,+}^{(k)} d_{j-1,-}^{(k)}}_{\substack{\text{interchange ✓} \\ -\text{ increases}}}. \end{aligned}$$

In summary, the dynamic is given by

$$d_{j,\pm}^{(k+1)} = d_{j,\pm}^{(k)} \pm (p_{j+1}^{(k)} d_{j+1,+}^{(k)} d_{j,-}^{(k)} - p_j^{(k)} d_{j,+}^{(k)} d_{j-1,-}^{(k)}). \tag{5.101}$$

Then, by (5.100) and $d_{j,+}^{(k)} + d_{j,-}^{(k)} = 1$ we get

$$d_{j,\pm}^{(k)} = \tfrac{1}{2}(1 \pm \bar{\theta}_j^{(k)}), \tag{5.102}$$

and consequently the recurrence (5.101) can be written in terms of $\bar{\theta}_j^{(k)}$ as

$$\bar{\theta}_j^{(k+1)} = \bar{\theta}_j^{(k)} + \tfrac{1}{2}(p_{j+1}^{(k)}(1+\bar{\theta}_{j+1}^{(k)})(1-\bar{\theta}_j^{(k)}) - p_j^{(k)}(1+\bar{\theta}_j^{(k)})(1-\bar{\theta}_{j-1}^{(k)})). \tag{5.103}$$

With [116] in mind, we declare

$$p_j^{(k)} = \frac{1}{2} \frac{\mu_h \wedge \mu_l}{d_{j,-}^{(k)} \mu_h + d_{j,+}^{(k)} \mu_l} \in [0, \tfrac{1}{2}], \tag{5.104}$$

where $a \wedge b \equiv \min\{a, b\}$ and also $a \vee b \equiv \max\{a, b\}$.

In the balanced case $\mu_h = \mu_l$ $(A = 0)$, we have $p_j^{(k)} = \frac{1}{2}$ independently of j, k. In the case of viscosity jump $\mu_h \neq \mu_l$ $(0 < |A| < 1)$, the probability of interchange at time $k+1$ depends on the relative position in terms of the mobility quotient $B = \mu_h/\mu_l$ (cf. sec. 5.6). For instance, when $\mu_h > \mu_l$ the lighter molecules rise through the heavier ones without many difficulties ($p_j^{(k)} \uparrow \frac{1}{2}$ as $\theta_j^{(k)} \uparrow 1$), whereas the molecules of the heavier fluid sink with lower speed because the fluid with phase $+$ has smaller mobility ($p_j^{(k)} \downarrow \frac{1}{2}B^{-1}$ as $\theta_j^{(k)} \downarrow -1$). The case $\mu_h < \mu_l$ follows analogously ($p_j^{(k)} \downarrow \frac{1}{2}B$ as $\theta_j^{(k)} \uparrow 1$ and $p_j^{(k)} \uparrow \frac{1}{2}$ as $\theta_j^{(k)} \downarrow -1$). A simple calculation yields

$$p_j^{(k)} = \frac{a}{1 - \bar{\theta}_j^{(k)} A} \quad \text{where} \quad a = \frac{\mu_h \wedge \mu_l}{\mu_h + \mu_l} = \frac{1 - |A|}{2} = \frac{1}{c_A^+ \vee c_A^-}. \tag{5.105}$$

Thus, if we scale the discretization as $r = ch$ for some $c > 0$, the recurrence (5.103) can be written as a finite difference equation

$$\frac{\bar\theta_j^{(k+1)} - \bar\theta_j^{(k)}}{\triangle t} = ca\left(\frac{(1+\bar\theta_{j+1}^{(k)})(1-\bar\theta_j^{(k)})}{1-\bar\theta_{j+1}^{(k)}A} - \frac{(1+\bar\theta_j^{(k)})(1-\bar\theta_{j-1}^{(k)})}{1-\bar\theta_j^{(k)}A}\right)\Big/\triangle x_2. \tag{5.106}$$

Notice that, by construction, there is no interchange of molecules outside $\{(t,x) : |x_2| < ct\}$. When $h \downarrow 0$, the scheme (5.106) converges formally to the Burgers type equation (5.28) where $\alpha = ca$ is the mixing speed. Since $0 < \alpha < 1$, necessarily

$$0 < c < a^{-1} = c_A^+ \vee c_A^-.$$

As we have mentioned, the aim of this stochastic process is just to give a simple way to outline the mixing phenomenon for the flat case. Similarly to the approach of Otto, while this random walk provides infinitely many trajectories $\theta_h = \{\theta_{s,j}^{(k)}\}$ starting from (5.98) (for different mixing speeds $0 < \alpha < 1$), the simulations evidence that $\theta_h \overset{*}{\rightharpoonup} \bar\theta_{A,\alpha}$. In other words, when $h \approx 0$, although each simulation yields a different picture, at the macroscopic level we cannot distinguish them. Moreover, $\bar\theta_{A,\alpha}$ can be (almost) recovered from each experiment separately by averaging it over lines as in Remark 5.2.2

$$\tfrac{1}{2N+1}\sum_{|s|\leq N}\theta_{s,j}^{(k)} \underset{N\to\infty}{\longrightarrow} \bar\theta_j^{(k)},$$

due to the central limit theorem.

5.6 The function Θ_A

Since the derivation of (5.14)(5.15) from [116] involves some parameters and computations, we have considered appropriate to give a brief explanation of it in order to save time to the reader. In [116] the phase "s" introduced by Otto takes values in $\{0,1\}$, while in this chapter the phase θ takes values in $\{-1,1\}$. Both are related via: $s = 0 \leftrightarrow \theta = 1$ and $s = 1 \leftrightarrow \theta = -1$. Thus, the density ρ and the **mobility** $m = \mu^{-1}$ are described in terms of the phase s as

$$a(t,x) = a^+ + (a^- - a^+)s(t,x), \quad a = \rho, m. \tag{5.107}$$

After rescaling in time, Otto considered the (normalized) IPM system

$$\partial_t s + \nabla\cdot(sw) = 0, \tag{5.108}$$

$$\mathrm{div} w = 0, \tag{5.109}$$

$$\mathrm{curl}((s/B + (1-s))w - si) = 0, \tag{5.110}$$

in $\mathbb{R}_+ \times \mathscr{D}$, starting from the unstable planar phase $s^\circ = \frac{1-\theta^\circ}{2}$ (5.6), where B is the **mobility quotient**

$$B = \frac{m_l}{m_h} = \frac{\mu_h}{\mu_l} = \frac{1+A}{1-A} > 0 \quad \leftrightarrow \quad A = \frac{B-1}{B+1} \in (-1,1).$$

Thus, one can easily check that (s,w) is a solution to (5.107)-(5.110) if and only if (θ, v) given by

$$\theta(t,x) = 1 - 2s(\alpha t, x), \qquad v(t,x) = \alpha w(\alpha t, x),$$

with $\alpha = 1 + 1/B$, solves IPM$_A$. After the relaxation explained in Appendix 5.5, Otto obtained the entropy solution

$$S_B(t,x) = \begin{cases} 0, & x_2 > Bt, \\ \frac{Bt - x_2}{Bt + (B-1)x_2 + \sqrt{B^2 t(Bt + (B-1)x_2)}}, & -t < x_2 < Bt, \\ 1, & -t > x_2, \end{cases}$$

of the scalar conservation law

$$\partial_t S + \partial_{x_2}\left(\frac{S(1-S)}{S + (1-S)/B}\right) = 0, \qquad S|_{t=0} = s^\circ.$$

Hence, since

$$\alpha = 1 + 1/B = c_A^-, \qquad B\alpha = 1 + B = c_A^+,$$

the function $\Theta_A(t,x) = 1 - 2S_B(\alpha t, x)$ is the entropy solution of the scalar conservation law (5.15). Clearly $\Theta_A(t,x) = \pm 1$ in $\Omega_\pm = \{(x,t) \in \mathbb{R}_+ \times \mathscr{D} \,:\, \pm x_2 > c_A^\pm t\}$. Inside the mixing zone $\Omega_{\mathrm{mix}} = \{(t,x) \in \mathbb{R}_+ \times \mathscr{D} \,:\, -c_A^- t < x_2 < c_A^+ t\}$, for $A = 0$ we have

$$\Theta_0(t,x) = \frac{x_2}{2t},$$

and for $0 < |A| < 1$ it is not difficult to check the following identities

$$\begin{aligned} \Theta_A(t,x) &= \frac{(x_2 - t) + \sqrt{Bt(t + Ax_2)}}{t + Ax_2 + \sqrt{Bt(t + Ax_2)}} \\ &= \frac{x_2 + At}{t + Ax_2 + \sqrt{(1 - A^2)t(t + Ax_2)}} \\ &= \frac{1}{A}\left(1 - \sqrt{\frac{(1 - A^2)t}{t + Ax_2}}\right). \end{aligned}$$

Proposition 5.6.1. *For $\mathscr{D} = \mathbb{R}^2$, Θ_A satisfies the following properties. At each time slice:*

(i) $\Theta_A(t,\cdot)$ *is continuous and smooth in* $\Omega_{\mathrm{mix}}(t)$*.*

(ii) $\Theta_A(t,\cdot)$ *is strictly x_2-increasing and concave (convex) for $A > 0$ ($A < 0$) in* $\Omega_{\mathrm{mix}}(t)$*.*

(iii) $\Theta_A(t,x) = \Theta_A(\tau, \frac{\tau}{t}x)$ *for all $\tau > 0$ and $x \in \mathbb{R}^2$.*

(iv) $\Theta_{-A}(t,x) = -\Theta_A(t,-x)$*.*

(v) For every $L = (l_1, l_2) \subset \alpha(-c_A^-, c_A^+)$*,*

$$\langle L \rangle_{A,\alpha} = \fint_L \Theta_A(\alpha, x_2)\,\mathrm{d}x_2 = \begin{cases} \frac{1}{A}\left(1 - \frac{2\sqrt{(1 - A^2)\alpha}}{\sqrt{\alpha + Al_1} + \sqrt{\alpha + Al_2}}\right), & A \neq 0, \\ \frac{l_1 + l_2}{4\alpha}, & A = 0. \end{cases}$$

For $\mathscr{D} = (-1,1)^2$ see section 5.6.1.

Proof. i is a straightforward computation. ii is a consequence of

$$\partial_{x_2}\Theta_A(t,x) = \tfrac{1}{2}\sqrt{(1-A^2)t}(t+Ax_2)^{-\frac{3}{2}} > 0,$$
$$\partial_{x_2}^2\Theta_A(t,x) = -\tfrac{3}{4}A\sqrt{(1-A^2)t}(t+Ax_2)^{-\frac{5}{2}}.$$

iiiiv follow from (5.14). v is due to, for $A=0$

$$\int \Theta_0(\alpha,x_2)\,\mathrm{d}x_2 = \frac{x_2^2}{4\alpha},$$

and for $A\neq 0$

$$\int \Theta_A(\alpha,x_2)\,\mathrm{d}x_2 = \frac{1}{A^2}\left(Ax_2 - 2\sqrt{(1-A^2)\alpha(\alpha+Ax_2)}\right).$$

□

Remark 5.6.1. To conclude we recall briefly the "uncertainty principle" presented in section 1.2. On the one hand, for $a=\rho,\mu$ given in terms of a Θ_A-mixing solution θ via (5.1), the Lebesgue differentiation theorem implies

$$\lim_{\substack{R\to\{x_0\}\\ \text{regular}}} \fint_R a(t,x)\,\mathrm{d}x = a(t,x_0),$$

for a.e. $x_0\in\mathscr{D}$ at each time slice $t\in\mathbb{R}_+$, where a jumps unpredictably between a_h and a_l on $\Omega_{\mathrm{mix}}(t)$ due to Thm. 5.2.1b. On the other hand, for every rectangle $R=S\times tL\subset\Omega_{\mathrm{mix}}(t)$ either large or close enough to the (space-time) boundary of the mixing zone, we have

$$\fint_R a(t,x)\,\mathrm{d}x \approx \tfrac{a_h+a_l}{2} + \tfrac{a_h-a_l}{2}\langle L\rangle_{A,\alpha},$$

at each time slice $t\in\mathbb{R}_+$, due to Thm. 5.2.1c. In other words, either the position is localized $\{x_0\}$ and so the phase is unpredictable or it is averaged in a suitable region R.

5.6.1 Transition to the stable planar phase

In this section we describe Θ_A in the confined domain $\mathscr{D}=(-1,1)^2$ once the mixing zone hits the lower or upper boundary. Immediately after the heavier fluid attains $x_2=-1$ ($c_A^-t>1$) the bottom of the tank begins to be filled up with it and the phases begin to separate

$$\Theta_A(t,x) = \begin{cases} \frac{x_2+At}{t+Ax_2+\sqrt{(1-A^2)t(t+Ax_2)}}, & d_A^-(t) < x_2 < 0,\\ +1, & d_A^-(t) > x_2,\end{cases}$$

and the same happens once the lighter one attains $x_2=1$ ($c_A^+t>1$)

$$\Theta_A(t,x) = \begin{cases} -1, & x_2 > d_A^+(t),\\ \frac{x_2+At}{t+Ax_2+\sqrt{(1-A^2)t(t+Ax_2)}}, & 0 < x_2 < d_A^+(t),\end{cases}$$

where $d_A^\pm$ are the free boundaries, to be determined.

By taking $\bar{v}_A = 0$ and $\bar{m}_A$ as in (5.27) ($\alpha = 1$), (5.18)(5.19) is automatically satisfied while (5.20) is equivalent to

$$[\Theta_A]_\pm \partial_t d_A^\pm = [\bar{m}_A]_\pm, \tag{5.111}$$

where $[\cdot]_\pm$ denotes the jump discontinuity at $x_2 = d_A^\pm$ respectively. By writing $d_A^\pm = \pm(1 - f_A^\pm)$, (5.111) turns out to be a Cauchy problem for $f_A^\pm$

$$\begin{aligned} \partial_t f_A^\pm &= F_A^\pm(t, f_A^\pm), \\ f_A^\pm|_{c_A^\pm t=1} &= 0, \end{aligned} \tag{5.112}$$

where

$$F_A^\pm(t, f) = \frac{1 \mp \Theta_A(t, \pm(1 - f(t)))}{1 - \Theta_A(t, \pm(1 - f(t)))A}.$$

By the Picard-Lindelöf Theorem, there is a unique solution to (5.112). Furthermore, it is strictly increasing with $f_A^\pm(t_A^\pm) = 1$ ($d_A^\pm(t_A^\pm) = 0$) at some $1 < c_A^\pm t_A^\pm < \infty$. Since (5.20) implies $\int \Theta_A(t, x)\,\mathrm{d}x = 0$ for all times, necessarily $t_A^\pm = t_A$. That is, the mixing zone collapses at this finite time t_A and the stable planar phase is reached. For $A = 0$ this is explicit

$$f_0(t) = 1 + 2t - 2\sqrt{2t},$$

for all $c_0^{-1} = \frac{1}{2} \leq t \leq 2 = t_0$.

Figure 5.7: On the left hand column we see a Matlab simulation ("solution") of this random walk stopped at some time starting from Fig. 5.6, while the right hand column shows the average over lines ("subsolution") of the previous picture. From top to bottom, the corresponding Atwood number A is $-\frac{1}{2}$, 0 and $\frac{1}{2}$ respectively (cf. Fig. 5.8).

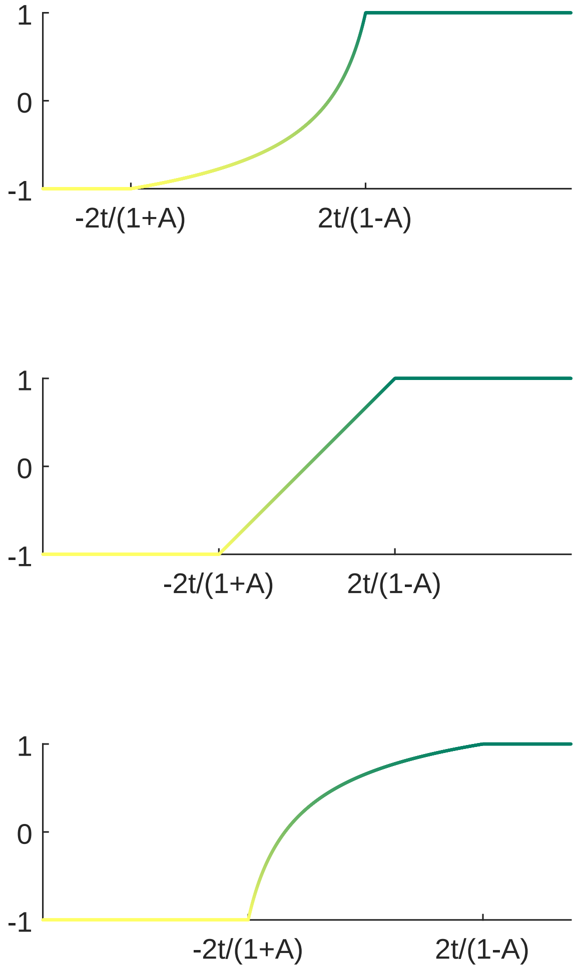


Figure 5.8: From top to bottom, we see the mixing profile $\Theta(t, x_2)$ at time $t = \frac{1}{2}$ for the Atwood number A equals $-\frac{1}{2}$, 0 and $\frac{1}{2}$ respectively.

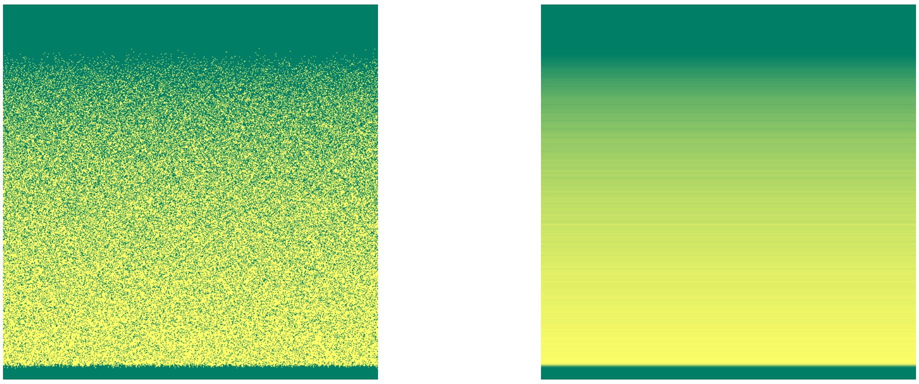


Figure 5.9: Evolution of Figure 5.7 for $A = -\frac{1}{2}$ at some $(c_A^-)^{-1} < t < (c_A^+)^{-1}$.

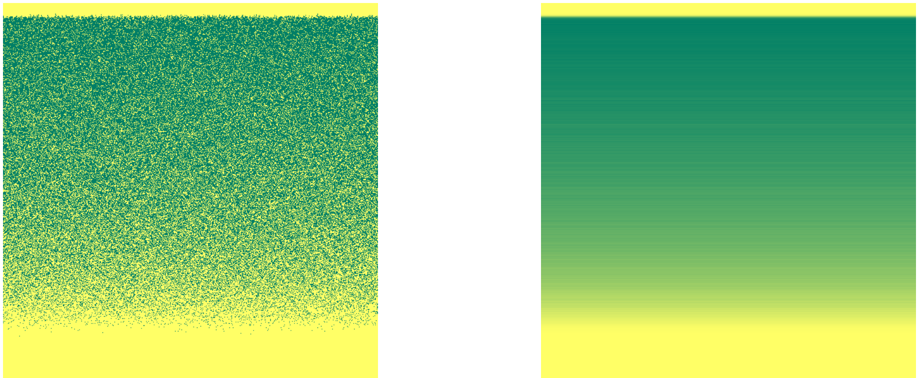


Figure 5.10: Evolution of Figure 5.7 for $A = \frac{1}{2}$ at some $(c_A^+)^{-1} < t < (c_A^-)^{-1}$.

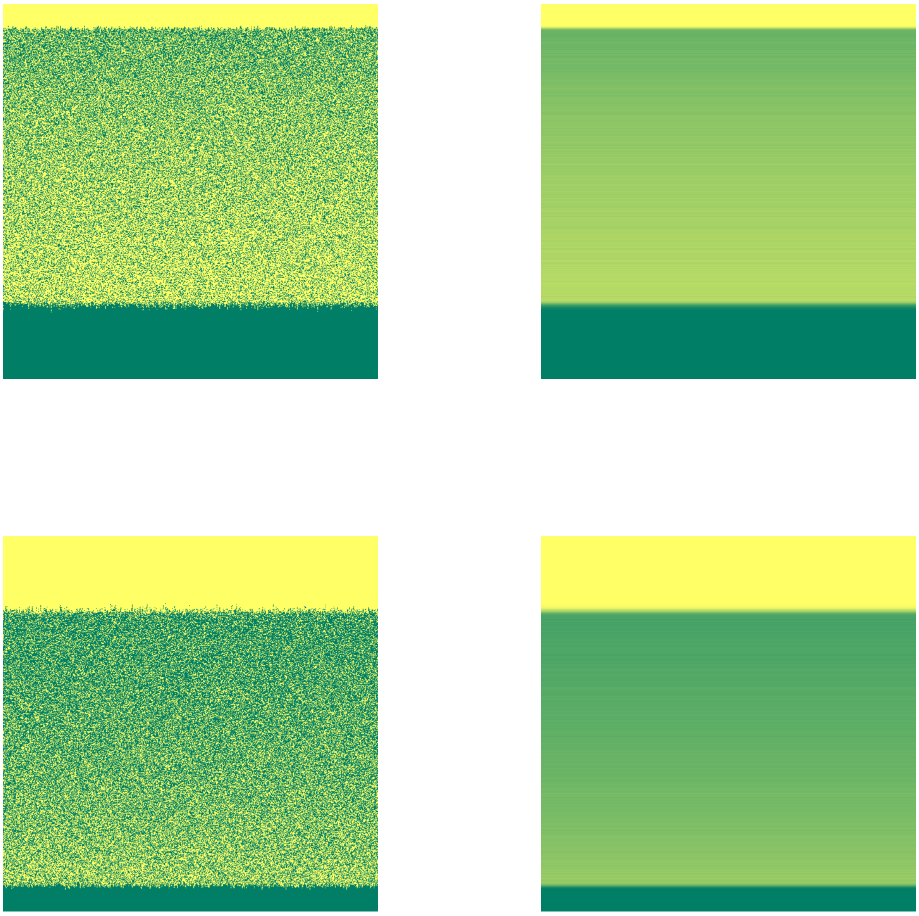


Figure 5.11: Evolution of Figure 5.7 for $A = -\frac{1}{2}, \frac{1}{2}$ at some $t > (c_A^-)^{-1} \vee (c_A^+)^{-1}$.

Chapter 6

Localized mixing zone for Muskat bubbles and turned interfaces

This chapter presents the paper [27], joint work with Ángel Castro and Daniel Faraco.

6.1 Introduction and main results

We consider two incompressible fluids with different constant densities ρ_-, ρ_+ and equal viscosity μ, separated by a connected curve $z^\circ = (z_1^\circ, z_2^\circ)$ inside a 2D porous medium with constant permeability κ (or Hele-Shaw cell [119]) and under the action of gravity $-\rho g i$. As we deal with closed and open curves, it is convenient to fix an orientation for z°. For closed curves we fix the clockwise orientation ($\circlearrowright$) and for open curves the orientation from $x_1 = -\infty$ to $+\infty$. Then, we denote Ω_-° (Ω_+°) by the domain to the left (right) side of z°. Thus, the initial density will be written as

$$\rho^\circ(x) := \begin{cases} \rho_-, & x \in \Omega_-^\circ, \\ \rho_+, & x \in \Omega_+^\circ, \end{cases} \tag{6.1}$$

for $x = (x_1, x_2) \in \mathbb{R}^2$. It is widely accepted that the dynamic of this two-phase flow can be modelled by the incompressible porous media (IPM) system

$$\partial_t \rho + \nabla \cdot (\rho v) = 0, \tag{6.2}$$

$$\nabla \cdot v = 0, \tag{6.3}$$

$$\tfrac{\mu}{\kappa} v = -\nabla p - \rho g i, \tag{6.4}$$

where $\rho(t, x) \equiv$ density, $v(t, x) \equiv$ velocity field, $p(t, x) \equiv$ pressure. By normalizing, we may assume w.l.o.g. that $|\rho_\pm| = \mu = \kappa = g = 1$.

The investigations on the Muskat problem ([110]) which deals with the interface evolution under the assumption of immiscibility, have been very intense both in the applied community due to the many applications (see e.g. [134, 131, 80, 97]) and in the theoretical side as this constitutes a challenging free boundary problem.

Mathematically, the theory has bifurcated into two regimes, the so-called stable regime and unstable regime. This division arises from the linear stability analysis of the equation for the interface evolution. It is classical (see e.g. [37]) that such linear stability is characterized by the sign of the Rayleigh-Taylor function $\sigma := (\rho_+ - \rho_-)\partial_\alpha z_1^\circ$ as follows:

$$\text{stable} \quad \text{on} \quad \sigma(\alpha) > 0, \tag{6.5a}$$

$$\text{unstable} \quad \text{on} \quad \sigma(\alpha) \leq 0. \tag{6.5b}$$

This simply classifies whether the heavier fluid remains (locally) below the lighter one or not. If the initial interface is a graph, $z^\circ(\alpha) = (\alpha, f^\circ(\alpha))$, the interface evolution is governed by a a nonlinear parabolic equation, which can be linearized as $\partial_t f = (\rho_+ - \rho_-)(-\Delta)^{1/2} f$. Hence, the stability simply depends on the sign of the density jump $\rho_+ - \rho_-$. Therefore, for $\rho_+ > \rho_-$ (i.e. the heavier fluid is below z°) what is called the fully stable regime, the analogy with the heat equation gives hope of well-posedness theory in a suitable Sobolev space H^k. We refer to the corresponding weak solutions to IPM as non-mixing solutions (see [136, 123, 7, 40, 37] for initial results). In the last years there have taken extensive steps to reduce the initial k (see [28, 35, 100, 2, 112]). The current world record is the result of Alazard and Nguyen [3] where they have proved the critical case $k = 3/2$ (see also [5, 4]). For small enough initial data these solutions are global-in-time. Additional results of global well-posedness for medium size initial data can be found in [32, 33] and global solutions with large initial slope in [41, 21, 57].

The instability in the linearization is called Rayleigh-Taylor (or Saffman-Taylor [119]) for the Muskat problem. In the graph case, it corresponds to $\rho_+ < \rho_-$ (i.e. the heavier fluid is above z°) what is called the fully unstable regime, and the analogy is now with the backwards heat equation. Therefore, it is to be expected that the problem is ill-posed unless the initial data is real-analytic C^ω. As a matter of fact, all the techniques available in the stable case catastrophic fail in this situation. Indeed, it can be proved that in the fully unstable regime, $\sigma(\alpha) \leq 0$ for every α, the Cauchy problem for f is ill-posed in Sobolev spaces (see e.g. [123]). However, practical and numerical experiments show the existence of the so-called mixing solutions, solutions in which there exist a mixing zone where the two fluids mix stochastically (see e.g. [134, 80]). Numerically, it can be seen that small disturbances of an analytic initial interface increases rapidly creating finger patterns at different scales in the unstable region (see e.g. [131, 97] and Figure 6.1).

In spite of the fact that the linearized problem is ill-posed and in accordance with what is observed in the experiments, weak solutions to IPM, in the fully unstable case, have been constructed in the last years by replacing the continuum free boundary assumption with the opening of a mixing zone Ω_{mix} where the fluids begin to mix indistinguishably. These mixing solutions (ρ, v) are recovered by the convex integration method applied in Ω_{mix} to a so-called "subsolution" $(\bar{\rho}, \bar{v}, \bar{m})$ (cf. section 6.2). These subsolutions are intended to be a kind of coarse-grained solutions to IPM, with $\bar{m}$ representing the relaxation of the momentum $\bar{\rho}\bar{v}$. The subsolutions are very related to the relaxed solutions appearing in the Lagrangian relaxation approach of Otto [116, 115] (see also [83]).

In the context of large data, an striking result from [24, 23] shows that there exist analytic initial interfaces in the fully stable regime (i.e. a graph) such that part of the curve turns to the unstable regime (i.e. no longer a graph) and later, at some $T_* > 0$, the interface $z(T_*)$ is analytic but at a point in the unstable region where it is not C^4. The argument in [23] could be adapted to prove weaker singularities in C^k where $k \geq 5$ (i.e. the interface leaves to be C^k but is still C^{k-1}). Thus, the Rayleigh-Taylor instability can arise spontaneously and the regularity might break down. After the blow-up time T_* it is to be expected that the Muskat problem is ill-posed.

In this chapter we give a method to construct mixing solutions to IPM in the Muskat partially unstable case. The original motivation was to continue the solutions after the breakdown described in the previous paragraph. However, there are numerous scenarios which are partially unstable. In this chapter we will concentrate on two of them: The so-called **bubble interfaces** where the two fluids are separated by a closed chord-arc curve (see [73] for the case with surface tension) and the **turned interfaces** where the interface is an open chord-arc curve which cannot be parametrized as a graph. We describe both scenarios readily, prior to the statement of the theorems.

The bubble type initial interfaces are described by

$$\begin{aligned} \Omega_-^\circ &\equiv \text{ exterior domain of } z^\circ, \\ \Omega_+^\circ &\equiv \text{ interior domain of } z^\circ, \end{aligned} \tag{6.6}$$

with $\rho_\pm = \pm 1$, for some closed chord-arc curve $z^\circ \in H^k(\mathbb{T};\mathbb{R}^2)$ with k big enough (cf. Figure 6.1(a)). Recall that we have taken z° clockwise oriented (↻) to be consistent with the notation in (6.1).

The turned type initial interfaces are described by

$$\begin{aligned} \Omega_-^\circ &\equiv \text{ upper domain of } z^\circ, \\ \Omega_+^\circ &\equiv \text{ lower domain of } z^\circ, \end{aligned} \tag{6.7}$$

with $\rho_\pm = \pm 1$, for some open chord-arc curve z° whose turned region $\{\partial_\alpha z_1^\circ(\alpha) \leq 0\}$ has positive measure. Here we consider both the x_1-periodic case $z^\circ - (\alpha, 0) \in H^k(\mathbb{T};\mathbb{R}^2)$ and the asymptotically flat case $z^\circ - (\alpha, 0) \in H^k(\mathbb{R};\mathbb{R}^2)$ with k big enough (cf. Figure 6.1(b)).

Now we are ready to state our two main theorems.

Theorem 6.1.1. *For every closed chord-arc curve $z^\circ \in H^6(\mathbb{T};\mathbb{R}^2)$ there exist infinitely many mixing solutions to IPM starting from* (6.1)(6.6) *with $\rho_\pm = \pm 1$.*

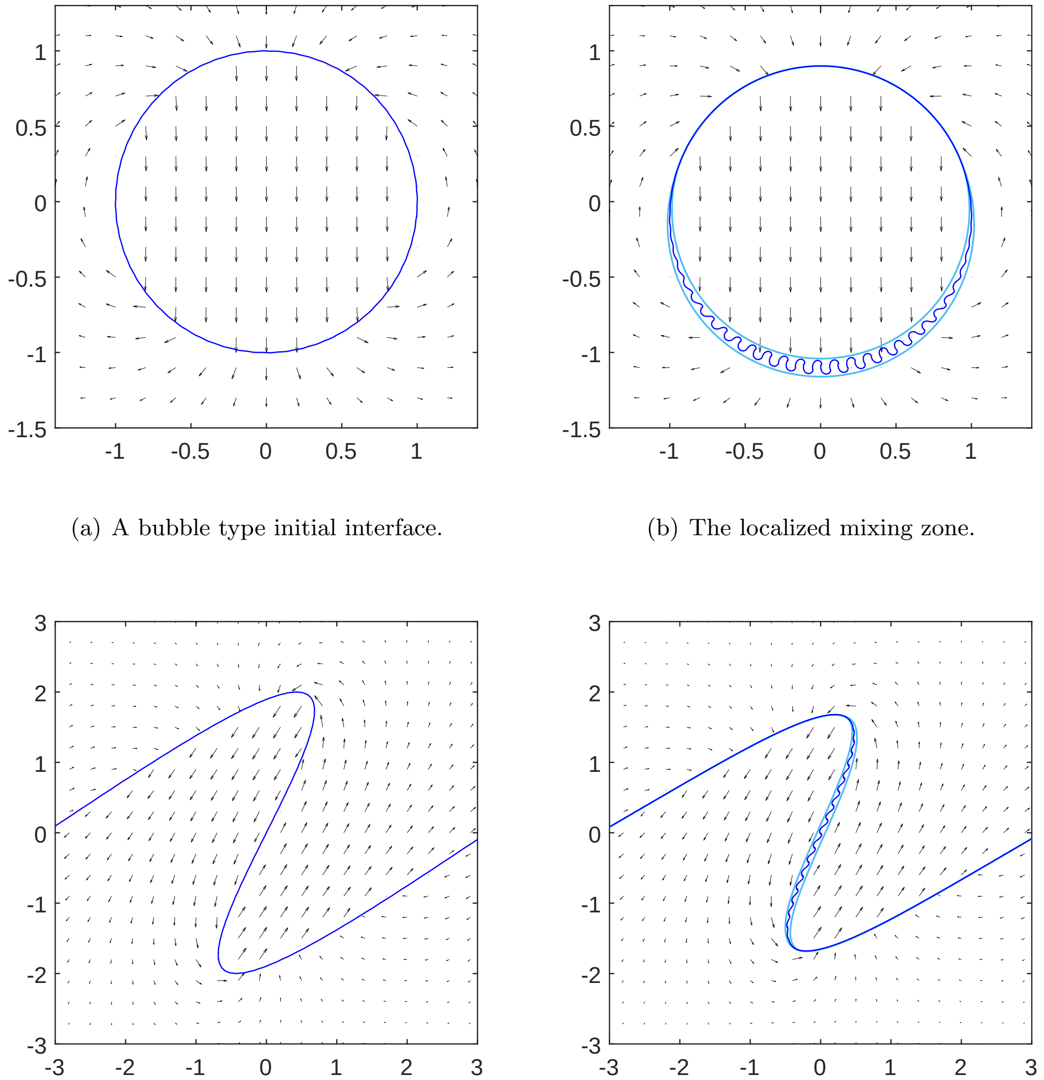
Theorem 6.1.2. *For every open chord-arc curve z°, either x_1-periodic $z^\circ - (\alpha, 0) \in H^6(\mathbb{T};\mathbb{R}^2)$ or asymptotically flat $z^\circ - (\alpha, 0) \in H^6(\mathbb{R};\mathbb{R}^2)$, whose turned region $\{\partial_\alpha z_1^\circ(\alpha) \leq 0\}$ has positive measure there exist infinitely many mixing solutions to IPM starting from* (6.1)(6.7) *with $\rho_\pm = \pm 1$.*

Remark 6.1.1. Theorem 6.1.2 is the first result proving the continuation of the evolution of IPM after the breakdown exhibited in [24, 23].

Remark 6.1.2. As in [126, 22, 70, 113], our mixing zone grows linearly in time around an evolving pseudo-interface. However, in Theorems 6.1.1 and 6.1.2 the mixing region must be localized in a neighborhood of the unstable region. Furthermore, this approach reveals the admissible regime for the growth-rate $c(\alpha)$ of the mixing zone compatible with the relaxation of IPM. This is

$$\left| c(\alpha) + \frac{\sigma(\alpha)}{\sqrt{\sigma(\alpha)^2 + \varpi(\alpha)^2}} \right| < 1 \quad \text{on} \quad c(\alpha) > 0, \tag{6.8}$$

which is characterized by the **Rayleigh-Taylor** function $\sigma := (\rho_+ - \rho_-)\partial_\alpha z_1^\circ$ and the **vorticity** strength $\varpi := -(\rho_+ - \rho_-)\partial_\alpha z_2^\circ$ along z° (cf. section 6.2). Observe that (6.8) prevents the two fluids from mixing wherever the initial interface is stable ($\sigma(\alpha) > 0$) and there is no vorticity ($\varpi(\alpha) = 0$).



(a) A bubble type initial interface.

(b) The localized mixing zone.

(c) A turned type initial interface.

(d) The localized mixing zone.

Figure 6.1: (a)(c) The initial interface $z^{\circ}(\alpha)$ separating two fluids with different constant densities $\rho_{\pm} = \pm 1$ as in (6.6)(6.7) respectively. (b)(d) At some $t > 0$, the two boundaries of the non-mixing zones $z_{\pm}(t,\alpha) = z(t,\alpha) \mp tc(\alpha)\tau(\alpha)^{\perp}$ (light blue) for some pseudo-interface $z(t,\alpha)$ and growth-rate $c(\alpha)$, with $\tau(\alpha) = \frac{\partial_\alpha z^{\circ}(\alpha)}{|\partial_\alpha z^{\circ}(\alpha)|}$. Inside the mixing zone $\Omega_{\mathrm{mix}}(t)$ we plot the Rayleigh-Taylor curve $z_{\mathrm{per}}(t)$ (dark blue) which starts from a tiny perturbation of z° (via the vortex-blob method). In all the figures we have added the coarse-grained velocity field $\bar{v}(t,x)$ outside Ω_{mix}.

In the context of modeling instabilities in Fluid Dynamics via convex integration, the first result in the IPM context (see also [39]) was proved in [126] where Székelyhidi constructed infinitely many weak solutions to IPM starting from the unstable planar interface. Remarkably, the coarse-grained density (the subsolution in the convex integration jargon) agrees with the Otto's Lagrangian relaxation of IPM (cf. [116] and also [102]). In [22] the first two authors and Córdoba constructed mixing solutions starting with a non-flat interface. In this work and all the subsequent ones, the mixing zone is described as an envelop of size $tc(\alpha)$ of a curve $z(t,\alpha)$ whose evolution is dictated by an operator which is an average of the classical Muskat operator. In [22] the coarse-grained density $\bar{\rho}$ is a continuous interpolation between the two fluids, which induces through an adapted h-principle a degraded mixing property ([26]). As a by-product of this version of the h-principle [26], one shows that the subsolution is recovered from the solution by taking suitable averages. Remarkably, if one considers instead piecewise constant coarse-grained densities, the evolution of the pseudo-interface greatly simplifies as was shown in [70] by Förster and Székelyhidi. See also [8, 113] for possible choices of the speed of opening of the mixing zone $c(\alpha)$.

After the works in IPM, instabilities for the incompressible Euler equations have been successfully modeled with related strategies, e.g. the Rayleigh-Taylor ([76, 75]) and the Kelvin-Helmholtz ([127, 103]) instabilities.

All the previous works deal either with the fully stable or fully unstable regime of the various instabilities and hence new twists should be added to the theory to deal with the partially unstable case. We finish the introduction with some comments on the natural obstructions and a non-technical description of the new viewpoints needed to address them. We believe that it is likely that the ideas from this chapter can be adapted and extended to consider different partially unstable scenarios in various problems concerning instabilities in Fluid Dynamics.

Since it is to be expected that the classical Muskat problem is ill-posed in this partially unstable situation, we need to see a way to find compatibility between the parabolic analysis for the stable case and the relaxation approach for the unstable case. In particular, the mixing region needs to envelope the unstable region. That is (recall $\sigma = (\rho_+ - \rho_-)\partial_\alpha z_1^\circ$)

$$\{\sigma(\alpha) \leq 0\} \subset \{c(\alpha) > 0\}. \tag{6.9}$$

As h-principles are by now standard [126, 22, 26], the main issue of the proof relies on building a mixing zone which admits a suitable subsolution $(\bar{\rho}, \bar{v}, \bar{m})$. We will follow [70] and declare $\bar{\rho}$ piecewise constant in the mixing zone. In fact, for the sake of simplicity during the introduction we will assume the simplest case, $\bar{\rho} = 0$ in Ω_{mix}.

At each time slice $0 < t \leq T \ll 1$, the mixing zone is the open set in $\mathbb{R}^2$ given by

$$\Omega_{\text{mix}}(t) := \{z_\lambda(t,\alpha) \,:\, c(\alpha) > 0,\, \lambda \in (-1,1)\}, \tag{6.10}$$

parametrized by the map

$$z_\lambda(t,\alpha) := z(t,\alpha) - \lambda t c(\alpha)\tau(\alpha)^\perp, \tag{6.11}$$

where $\tau(\alpha)$ is an unitary vector field, $c(\alpha)$ is the growth-rate of the mixing zone and $z(t,\alpha)$ is the pseudo-interface evolving from $z^\circ(\alpha)$, that we have to determine.

In order to optimize the speed of opening of the mixing zone, it is convenient to take τ as the tangential vector field to z°

$$\tau(\alpha) = \operatorname{sgn}(\rho_+ - \rho_-)\frac{\partial_\alpha z^\circ(\alpha)}{|\partial_\alpha z^\circ(\alpha)|}. \tag{6.12}$$

With our ansatz for $\bar{\rho}$ as in [70] and this optimal choice for τ, the admissible regime for $c(\alpha)$ compatible with the relaxation of IPM becomes

$$\left|2c(\alpha)+\frac{\sigma(\alpha)}{\sqrt{\sigma(\alpha)^2+\varpi(\alpha)^2}}\right|<1 \quad \text{on} \quad c(\alpha)>0. \tag{6.13}$$

We remark in passing that $2c(\alpha)$ above can be replaced by $\frac{2N}{2N-1}c(\alpha)$ for any $N \geq 1$ as in [70, 113], which yields (6.8) as $N \to \infty$ (cf. section 6.6). Observe that this inequality requires $c(\alpha) = 0$ if $\sigma(\alpha) = |(\sigma(\alpha), \varpi(\alpha))|$, or equivalently $\partial_\alpha z_1^\circ(\alpha) = \operatorname{sgn}(\rho_+ - \rho_-)|\partial_\alpha z^\circ(\alpha)|$ (cf. Remark 6.1.2). Since in the regimes we are considering there are always such points, we are forced to treat the case where there is no opening in some region, i.e. $c(\alpha) = 0$. An extra difficulty at this level is that our estimates need certain smoothness in c (i.e. the very definition of the velocity) which necessarily creates cusp singularities on Ω_{mix}. We deal with this problem by interpreting the mixing zone as a superposition of regular domains (cf. Figure 6.2 and Lemma 6.2.1).

Next we turn to the coarse-grained velocity and the associated Muskat type operator. Here we start from [70] as we have chosen the same ansatz for the coarse-grained density and then explain the new idea. The Förster-Székelyhidi's velocity is also an average of the classical Muskat velocity as in [22] but only between the two boundaries of the non-mixing zones $z_\pm = z \mp tc\tau^\perp$. The associated Muskat type operator is (cf. section 6.2)

$$B := \frac{1}{2}\sum_{a=\pm} B_a, \qquad B_a := \sum_{b=\pm} B_{a,b}, \tag{6.14}$$

where

$$B_{a,b}(t,\alpha) := \frac{\rho_+ - \rho_-}{4\pi}\int \left(\frac{1}{z_a(t,\alpha)-z_b(t,\beta)}\right)_1 (\partial_\alpha z_a(t,\alpha) - \partial_\alpha z_b(t,\beta))\,\mathrm{d}\beta. \tag{6.15}$$

We remark in passing that, for open curves as in Theorem 6.1.2, all these integrals are taken with the Cauchy's principal value at infinity. However, we will focus on the closed case until section 6.6 for clarity of exposition.

The evolution of z is driven by the operator B. On the one hand, as it is explained in the discussion after (6.13), in the partially unstable case there is always a non-mixing region where we must solve a classical Muskat equation exactly

$$\partial_t z = B \quad \text{on} \quad c(\alpha) = 0. \tag{6.16}$$

On the other hand, the flexibility of the notion of subsolution gives some space to define the pseudo-interface ([70, 113]). Namely, in the mixing region it is enough to solve (6.16) approximately

$$\partial_t z = B + \text{error} \quad \text{on} \quad c(\alpha) > 0,$$

where the error must be small in some sense that shall be specified in sections 6.2 and 6.2.2. Due to the Rayleigh-Taylor instability, it is to be expected that the choice error $= 0$ above yields an ill-posed equation as in the fully unstable regime. In spite of this, following another clever idea from [70], in the fully unstable regime it is possible to take error $= B^{1)} - B +$ error, where $B^{1)}$ denotes the first order expansion in time of B. This choice yields the following well-defined evolution for z

$$\partial_t z = B^{1)} + \text{error} \quad \text{on} \quad c(\alpha) > 0. \tag{6.17}$$

We remark that, if the error in (6.17) was zero, then the equations (6.16) and (6.17) do not match at $c(\alpha) = 0$. In order to glue these equations we first introduce a partition of the unity $\{\psi_0, \psi_1\}$ which, as required in (6.9), allows also to open the mixing zone slightly inside the stable region, namely $\operatorname{supp}\psi_0 \subset \{\partial_\alpha z_1^\circ(\alpha) > 0\}$ and $\operatorname{supp}\psi_1 = \operatorname{supp} c$. That is, we bypass the gluing problem by writing

$$\partial_t z = \psi_0 B + \psi_1 B^{1)} + \text{error} \quad \text{on} \quad \mathbb{T},$$

where the error is supported on $\{c(\alpha) > 0\}$. Yet the energy inequalities that we obtain for the operator $\psi_0 B$ (or other modifications) yields a factor $1/c$ which blows up in the region where $c(\alpha)$ tends to zero. The way out of this vicious circle is to treat the interaction between separate boundaries as a perturbation. In this way, one can write $B = E + \text{error}$ in such a way that E yields good energy inequalities and the error is small in the supremum norm and supported on $\{c(\alpha) > 0\}$. Thus, the perturbation can be absorbed in the relaxation even if its derivatives are badly behaving. Hence, we will solve

$$\partial_t z = \psi_0 E + \psi_1 E^{1)} + \text{error} \quad \text{on} \quad \mathbb{T},$$

for some error term supported on $\{c(\alpha) > 0\}$, where $E^{1)}$ denotes the first order expansion in time of E. Essentially, $E = B_{+,+} + B_{-,-}$ as the factor $1/c$ comes from the terms with $a \neq b$ in (6.15).

Organization of the chapter. We start section 2 by recalling briefly the Classical and the Mixing Muskat problem. After this, we recall also the concepts of mixing solution and subsolution, as well as the h-principle in IPM. Then, we define our ansatz for the subsolution in terms of the mixing zone and derive the conditions for the growth-rate c and the pseudo-interface z under which such subsolution truly exists. The construction of a pair (c, z) satisfying such requirements appears in sections 6.3-6.5. Finally, we prove in section 6.6 the Theorems 6.1.1, 6.1.2 and the optimal regime for c given in (6.8).

6.2 The mixing zone and the subsolution

The Muskat Problem

The Muskat problem describes IPM under the assumption that there is a time-dependent oriented curve $z(t, \alpha)$ separating $\mathbb{R}^2$ into two complementary open domains

$$\begin{aligned} \Omega_-(t) &\equiv \text{ domain to the left side of } z(t), \\ \Omega_+(t) &\equiv \text{ domain to the right side of } z(t), \end{aligned} \tag{6.18}$$

each one occupied by a fluid with different constant densities ρ_- and ρ_+ respectively.

The incompressibility condition (6.3) implies that $v = \nabla^\perp \psi$ for some stream function $\psi(t, x)$. Hence, Darcy's law (6.4) can be written in complex coordinates as $\nabla(p + i\psi) = -i\rho$, which yields the following Poisson equation ($\nabla^* \nabla = \Delta$)

$$\Delta(p + i\psi) = -i\nabla^* \rho.$$

In view of (6.18), the density jump along z implies that

$$\nabla\rho = -(\rho_+ - \rho_-)\partial_\alpha z^\perp \delta_z,$$

in the sense of distributions. Hence, p and ψ are recovered from the Poisson equation through the Newtonian potential

$$(p+i\psi)(t,x)=\frac{\rho_+-\rho_-}{2\pi}\int\log|x-z(t,\beta)|\partial_\alpha z(t,\beta)^*\,\mathrm{d}\beta,\qquad x\neq z(t,\beta).$$

Then, p and ψ are continuous but have discontinuous gradients along z, and indeed $\Delta(p+i\psi)=(\sigma+i\varpi)\delta_z$ where $\sigma\equiv$ Rayleigh-Taylor and $\varpi\equiv$ vorticity strength ($\Delta\psi=\nabla^\perp\cdot v=\omega$), which satisfy

$$\sigma+i\varpi=(\rho_+-\rho_-)\partial_\alpha z^*. \tag{6.19}$$

The velocity v is recovered from the vorticity through the Biot-Savart law

$$\begin{aligned} v(t,x)&=\left(\frac{1}{2\pi i}\int\frac{\varpi(t,\beta)}{x-z(t,\beta)}\,\mathrm{d}\beta\right)^*\\ &=-\frac{\rho_+-\rho_-}{2\pi}\int\left(\frac{1}{x-z(t,\beta)}\right)_1\partial_\alpha z(t,\beta)\,\mathrm{d}\beta,\qquad x\neq z(t,\beta),\end{aligned} \tag{6.20}$$

where we have applied that $\varpi=-(\rho_+-\rho_-)\partial_\alpha z_2$ and the Cauchy's argument principle in the last equality

$$\left(\int\frac{\partial_\alpha z(t,\beta)}{x-z(t,\beta)}\,\mathrm{d}\beta\right)_1=0,\qquad x\neq z(t,\beta). \tag{6.21}$$

It is easy to see that v is bounded, smooth outside z but with tangential discontinuities along z. Its normal component is well defined and satisfies

$$\lim_{\Omega_\pm(t)\ni x\to z(t,\alpha)}(v(t,x)-B(t,\alpha))\cdot\partial_\alpha z(t,\alpha)^\perp=0,$$

where

$$B(t,\alpha):=\frac{\rho_+-\rho_-}{2\pi}\int\left(\frac{1}{z(t,\alpha)-z(t,\beta)}\right)_1(\partial_\alpha z(t,\alpha)-\partial_\alpha z(t,\beta))\,\mathrm{d}\beta. \tag{6.22}$$

Observe that the operator B is obtained by adding a suitable tangential term to the velocity (6.20) and then taking the limit $\Omega_\pm(t)\ni x\to z(t,\alpha)$. We refer to (6.22) as the classical Muskat operator. Let us remark that this operator (6.22) coincides with (6.14) when tc is identically zero. Since it only appears in this subsection 6.2 and the notation is heavy enough, we do not give it another name.

Finally, it is easy to check that the conservation of mass equation (6.2) is equivalent to find z satisfying

$$(\partial_t z-B)\cdot\partial_\alpha z^\perp=0. \tag{6.23}$$

Thus, the Muskat problem is equivalent to solve this Cauchy problem for the interface z starting from z° given in (6.16). We remark that because of (6.23), one may add any tangential term to (6.16). This only changes the parametrization and does not modify the geometric evolution of the curve. We refer to (6.16) as the Classical Muskat problem.

Assuming that the interface can be parametrized as a graph, $z(t,\alpha)=\alpha+if(t,\alpha)$ in complex coordinates, the equation (6.16) reads as

$$\partial_t f = \frac{\rho_+ - \rho_-}{2\pi}\mathrm{pv}\int_{\mathbb{R}}\left(\frac{1}{1+i\Delta_\beta f}\right)_1 \partial_\alpha\Delta_\beta f\,\mathrm{d}\beta,$$

which can be linearized as $\partial_t f = (\rho_+ - \rho_-)(-\Delta)^{1/2} f$. In analogy with the heat equation, the fully stable regime ($\rho_+ > \rho_-$) admits a parabolic analysis through energy estimates.

However, the same strategy for the fully unstable regime ($\rho_+ < \rho_-$) is not viable. Despite this, mixing solutions to IPM starting from fully unstable Muskat inital data have been constructed in the last years through the convex integration method [126, 22, 70, 113]. In these works, the mixing zone is given as in (6.10)(6.11) but with $\tau = (-1,0)$ instead of (6.12). More generally, we may consider any unitary vector field $\tau(\alpha)$ satisfying

$$(\rho_+ - \rho_-)\partial_\alpha z^\circ(\alpha)\cdot\tau(\alpha) > 0 \quad \text{on} \quad c(\alpha) > 0.$$

Thus, the triplet (τ, c, z) parametrizes the mixing zone, which does not exist when $tc(\alpha)=0$. Here we follow [70, 113], where $\bar\rho = 0$ on Ω_{mix}. In this case, the coarse-grained velocity becomes

$$\bar v(t,x) = -\frac{\rho_+ - \rho_-}{4\pi}\sum_{b=\pm}\int\left(\frac{1}{x - z_b(t,\beta)}\right)_1 \partial_\alpha z_b(t,\beta)\,\mathrm{d}\beta, \qquad x \neq z_b(t,\beta),$$

where $z_\pm(t,\alpha) = z(t,\alpha) \mp tc(\alpha)\tau(\alpha)^\perp$ are the two boundaries of the non-mixing zones. The admissible regime for $c(\alpha)$ compatible with the relaxation of IPM is

$$\left|2c(\alpha) + \frac{\sigma(\alpha)}{(\rho_+ - \rho_-)\partial_\alpha z^\circ(\alpha)\cdot\tau(\alpha)}\right| < 1 \quad \text{on} \quad c(\alpha) > 0, \tag{6.24}$$

which agrees with [70, 113] as in this case $\rho_\pm = \mp 1$, $\partial_\alpha z^\circ = (1, \partial_\alpha f^\circ)$ and $\tau = (-1, 0)$ (cf. Rem. 6.2.4). Observe that (6.24) requires $c(\alpha) = 0$ if $\partial_\alpha z_1^\circ(\alpha) = \partial_\alpha z^\circ(\alpha)\cdot\tau(\alpha)$. In view of (6.9), this prevents some choices for $\tau(\alpha)$ as for instance the one from [22, 70, 26]. Thus, we really need to optimize by opening the mixing zone perpendicularly to the curve (this is also the case in [103]). This is why we have chosen τ as in (6.12). With such optimal choice for τ, (6.24) reads as (6.13) (recall (6.19)).

Once $\tau(\alpha)$ and $c(\alpha)$ are fixed, we must determine the time-dependent pseudo-interface $z(t,\alpha)$. Modulo technical details which will be explained in section 6.2.2, the existence of a relaxed momentum $\bar m(t,x)$ is reduced to find z satisfying

$$\int_0^\alpha \Big((\partial_t z - B)\cdot\partial_\alpha z^\perp + tD\cdot\partial_\alpha(c\tau)\Big)\,\mathrm{d}\alpha' = o(t)c(\alpha), \tag{6.25}$$

uniformly in α as $t \to 0$, where

$$D(t,\alpha) := -\frac{1}{2}\sum_{a=\pm} aB_a - i(c\tau + \tfrac{1}{2}), \tag{6.26}$$

with τ, c, B and B_a given in (6.10)-(6.15). Observe that $D\cdot\partial_\alpha(c\tau) = 0$ for $\partial_\alpha z = (1, \partial_\alpha f)$ and $\tau = (-1, 0)$, and thus it does not appear in [70, 113]. Hence, the equation (6.25) generalizes both (6.16) and (6.17). As we mentioned in the introduction, we cannot simply glue these evolution equations because they do not match at $c(\alpha) = 0$. In order to interpolate between the two regions, we introduce a partition of the unity $\{\psi_0, \psi_1\}$ subordinated to $\{\partial_\alpha z_1^\circ(\alpha) > 0\}$ and $\{c(\alpha) > 0\}$ respectively. Then, we consider (cf. (6.61))

$$\partial_t z = \psi_0 E + \psi_1 E^{1)} + \text{error}.$$

Here, E should be an extension of B and good for energy inequalities, which justify its name twice.

In view of (6.16) and (6.17), one would be initially tempted to take $E = B$. However, the terms with $a \neq b$ in (6.14) introduce a factor $\partial_\alpha \log c(\alpha)$ in the energy estimates which we did not see how to compensate. Thus, we will declare

$$E = \sum_{b=\pm} B_{b,b}, \tag{6.27}$$

which equals B on $tc(\alpha) = 0$ and only includes interaction of stable Muskat type.

The error term is localized on the mixing region with order t. This is

$$\text{error} = -(t\kappa + i(tD^{(0)} \cdot \partial_\alpha(c\tau) + h\psi_1)\partial_\alpha z^\circ),$$

where $D^{(0)} = D|_{t=0}$, $\kappa = \partial_t(E - B)|_{t=0}$ depends on the initial curvature and $h = O(t^2)$ is a time-dependent average. As a result, $D^{(0)}$ and κ only depends on z° while $h(t)$ depends on $z(t)$ but not on α. This allows to treat the error as a harmless term in the energy estimates.

Weak solutions, subsolutions and the mixing zone

Let us start by recalling the rigorous definition of weak solutions, mixing solutions and subsolutions in the IPM context.

Given $T > 0$ and ρ° as in (6.1), a **weak solution** to IPM

$$(\rho, v) \in C([0, T]; L^\infty_{w^*}(\mathbb{R}^2; [-1, 1] \times \mathbb{R}^2))$$

satisfies that, for every test function $\phi \in C^1_c(\mathbb{R}^3)$ with $\phi^\circ := \phi|_{t=0}$ and $0 < t \leq T$:

$$\int_0^t \int_{\mathbb{R}^2} \rho(\partial_t \phi + v \cdot \nabla\phi)\,\mathrm{d}x\,\mathrm{d}s = \int_{\mathbb{R}^2} \rho(t)\phi(t)\,\mathrm{d}x - \int_{\mathbb{R}^2} \rho^\circ \phi^\circ\,\mathrm{d}x, \tag{6.28a}$$

$$\int_0^t \int_{\mathbb{R}^2} v \cdot \nabla\phi\,\mathrm{d}x\,\mathrm{d}s = 0, \tag{6.28b}$$

$$\int_0^t \int_{\mathbb{R}^2} (v + \rho i) \cdot \nabla^\perp \phi\,\mathrm{d}x\,\mathrm{d}s = 0. \tag{6.28c}$$

In addition, a weak solution is a **mixing solution** if, at each $0 < t \leq T$, the space $\mathbb{R}^2$ is split into three complementary open domains, $\Omega_+(t)$, $\Omega_-(t)$ and $\Omega_{\text{mix}}(t)$, satisfying that (ρ, v) is continuous on the non-mixing zones $\Omega_\pm$:

$$\rho = \pm 1 \quad \text{on} \quad \Omega_\pm, \tag{6.29}$$

while it behaves wildly inside the mixing zone Ω_{mix}:

$$\int_\Omega (1 - \rho^2)\,\mathrm{d}x = 0 < \int_\Omega (1 - \rho)\,\mathrm{d}x \int_\Omega (1 + \rho)\,\mathrm{d}x, \tag{6.30}$$

for every open $\emptyset \neq \Omega \subset \Omega_{\mathrm{mix}}(t)$.
Conversely, we say that (ρ, v) is a **non-mixing solution** if $\Omega_{\mathrm{mix}} = \emptyset$.

In convex integration, a subsolution (a macroscopic solution) is defined in term of a conservation law and a relaxed constitutive relation, which is typically given by the Λ-convex hull. In the IPM context, the hull was computed in [126] (see also [102] for related computations).

Given $T > 0$ and ρ° as in (6.1), a **subsolution** to IPM

$$(\bar{\rho}, \bar{v}, \bar{m}) \in C([0,T]; L^\infty_{w^*}(\mathbb{R}^2; [-1,1] \times \mathbb{R}^2 \times \mathbb{R}^2))$$

satisfies that, for every test function $\phi \in C^1_c(\mathbb{R}^3)$ with $\phi^\circ := \phi|_{t=0}$ and $0 < t \leq T$:

$$\int_0^t \int_{\mathbb{R}^2} (\bar{\rho}\partial_t\phi + \bar{m} \cdot \nabla\phi) \,\mathrm{d}x \,\mathrm{d}s = \int_{\mathbb{R}^2} \bar{\rho}(t)\phi(t) \,\mathrm{d}x - \int_{\mathbb{R}^2} \rho^\circ \phi^\circ \,\mathrm{d}x, \tag{6.31a}$$

$$\int_0^t \int_{\mathbb{R}^2} \bar{v} \cdot \nabla\phi \,\mathrm{d}x \,\mathrm{d}s = 0, \tag{6.31b}$$

$$\int_0^t \int_{\mathbb{R}^2} (\bar{v} + \bar{\rho} i) \cdot \nabla^\perp\phi \,\mathrm{d}x \,\mathrm{d}s = 0, \tag{6.31c}$$

such that, at each $0 < t \leq T$, the space $\mathbb{R}^2$ is split into three complementary open domains, $\Omega_+(t)$, $\Omega_-(t)$ and $\Omega_{\mathrm{mix}}(t)$ satisfying that

$$\bar{\rho} = \pm 1, \quad \bar{m} = \bar{\rho}\bar{v} \quad \text{on} \quad \Omega_\pm, \tag{6.32a}$$

$$|2(\bar{m} - \bar{\rho}\bar{v}) + (1 - \bar{\rho}^2)i| < (1 - \bar{\rho}^2) \quad \text{on} \quad \Omega_{\mathrm{mix}}. \tag{6.32b}$$

In addition, it is required that

$$\sup_{0 \leq t \leq T} \|\bar{v}(t)\|_{L^\infty} < \infty. \tag{6.33}$$

Remark 6.2.1. Notice that the pressure does not appear in (6.28c)(6.31c). For completeness we will show in Lemma 6.7.1 how $p \in C([0,T] \times \mathbb{R}^2)$ is recovered and its relation with $\bar{p}$. We are not aware of similar computations for the IPM pressure in the convex integration framework.

Theorem 6.2.1 (H-principle in IPM)**.** *Assume that there exists a subsolution $(\bar{\rho}, \bar{v}, \bar{m})$ to IPM starting from ρ°, for some $T > 0$, $\Omega_\pm$ and Ω_{mix}. Then, there exist infinitely many mixing solutions (ρ, v) to IPM starting from ρ°, for the same $T > 0$, $\Omega_\pm$ and Ω_{mix}, and satisfying $(\rho, v) = (\bar{\rho}, \bar{v})$ outside Ω_{mix}.*

The proof of this h-principle for the $L^\infty_{t,x}$ case can be found in [126], and the generalization to $C_t L^\infty_{w^*}$ follows from chapter 5. As noticed in [126], the inequality (6.32b) only provides solutions in L^2. Remarkably, Székelyhidi computed in [126, Prop. 2.4] the additional inequalities which yield solutions in L^∞. Recall that it is checked in chapter 5 that, if $\bar{v}$ is controlled as in (6.33), then these additional inequalities are automatically satisfied.

By Theorem 6.2.1, the construction of mixing solutions as stated in Theorems 6.1.1 and 6.1.2 is reduced to constructing suitable subsolutions $(\bar{\rho}, \bar{v}, \bar{m})$ adapted to Ω_{mix}.

As described in the intro, it follows from (6.10)-(6.12) that the mixing zone is prescribed by the **growth-rate** c and the **pseudo-interface** z. With this terminology, for bubble interfaces (6.6) we define

$$\begin{aligned}\Omega_-(t) &\equiv \text{ exterior domain of } z_-(t),\\ \Omega_+(t) &\equiv \text{ interior domain of } z_+(t),\end{aligned} \tag{6.34}$$

and for turned interfaces (6.7)

$$\begin{aligned}\Omega_-(t) &\equiv \text{ upper domain of } z_-(t),\\ \Omega_+(t) &\equiv \text{ lower domain of } z_+(t).\end{aligned} \tag{6.35}$$

Thus, $\Omega_+(t)$, $\Omega_-(t)$ and $\Omega_{\text{mix}}(t)$ are complementary open domains in $\mathbb{R}^2$. For each region $r = +, -, \text{mix}$, we denote $\Omega_r := \{(t,x) \,:\, x \in \Omega_r(t),\, 0 \leq t \leq T\}$.

Remark 6.2.2. For the sake of simplicity we will consider from now on the closed case (6.34) and go back at section 6.6 with the open case (6.35). Recall that for closed interfaces we have assumed that z° is clockwise oriented (↻). In addition, we may assume w.l.o.g. that z° is the arc-length ($|\partial_\alpha z^\circ| = 1$) parametrization, although $z(t)$ will not be it in general. Thus, we fix $\mathbb{T} = [-\ell_\circ/2, \ell_\circ/2]$ where $\ell_\circ := \text{length}(z^\circ)$.

For a general pair (c,z) we construct in the next section 6.2.1 a suitable triplet $(\bar{\rho}, \bar{v}, \bar{m})$ adapted to Ω_{mix}. After this, we derive in section 6.2.2 conditions for (c,z) under which this $(\bar{\rho}, \bar{v}, \bar{m})$ becomes a subsolution. Finally, we will prove in sections 6.3-6.5 the existence of a pair (c,z) satisfying such requirements.

Hypothesis on (c,z). Along the rest of this section 6.2 we will assume the existence of $\delta, T > 0$ such that

$$c \in C^{1,\delta}(\mathbb{T}), \qquad c \geq 0, \tag{6.36}$$

and

$$z \in C^1([0,T]; C^{1,\delta}(\mathbb{T};\mathbb{R}^2)), \qquad z|_{t=0} = z^\circ \in C^{2,\delta}(\mathbb{T};\mathbb{R}^2), \tag{6.37}$$

satisfying the following equi-angle condition

$$\mathcal{A}(c,z) := \inf\left\{ \frac{\partial_\alpha z_\lambda(t,\alpha)}{|\partial_\alpha z_\lambda(t,\alpha)|} \cdot \tau(\alpha) \,:\, \alpha \in \mathbb{T},\, \lambda \in [-1,1],\, 0 \leq t \leq T \right\} > 0, \tag{6.38}$$

and the following equi-chord-arc condition

$$\mathcal{C}(c,z) := \sup\left\{ \frac{\sqrt{\beta^2 + ((\lambda-\mu)tc(\alpha))^2}}{|z_\lambda(t,\alpha) - z_\mu(t,\alpha-\beta)|} \,:\, \alpha,\beta \in \mathbb{T},\, \lambda,\mu \in [-1,1],\, 0 \leq t \leq T \right\} < \infty, \tag{6.39}$$

where we recall that $z_\lambda = z - \lambda t c \tau^\perp$ with $\tau = \partial_\alpha z^\circ$.

The condition (6.38) controls the angle between the family of curves z_λ w.r.t. τ. The equi-chord-arc condition (6.39) bounds the singularity due to the denominator of the operators $B_{a,b}$ (6.15), while the numerator justifies the regularity assumptions (6.36)(6.37). In addition, all they have the following useful consequence.

Remark 6.2.3. The conditions (6.36)-(6.39) imply that map $(\alpha, \lambda) \mapsto z_\lambda(t, \alpha)$ is a diffeomorphism from $\{c(\alpha) > 0\} \times (-1, 1)$ to $\Omega_{\text{mix}}(t)$ with Jacobian $tc(\partial_\alpha z_\lambda \cdot \tau) > 0$.

In section 6.3 we will construct a suitable smooth growth-rate c. Once c is fixed, we will still assume (6.37)-(6.39) in section 6.4. Finally, we will construct a time-dependent pseudo-interface z satisfying such conditions in section 6.5.

We conclude this subsection recalling the auxiliary lemma 6.2.1 which allows to integrate by parts, under certain conditions, on the domain with cusp singularities Ω_{mix}.

Lemma 6.2.1. *Fix $0 \le t \le T$. Let $f \in L^\infty(\mathbb{R}^2) \cap C^1(\mathbb{R}^2 \setminus (z_+(t) \cup z_-(t)))$ satisfying $\nabla \cdot f = 0$ outside $z_+(t) \cup z_-(t)$ and with well-defined continuous limits*

$$\begin{aligned} f_a^a(\alpha) &:= \lim_{\Omega_a(t) \ni x \to z_a(t,\alpha)} f(x), \\ f_a^{\text{mix}}(\alpha) &:= \lim_{\Omega_{\text{mix}}(t) \ni x \to z_a(t,\alpha)} f(x), \qquad tc(\alpha) > 0, \end{aligned} \tag{6.40}$$

whenever $z_a(t, \alpha) \in \partial\Omega_r(t)$ for $a = \pm$ and $r = +, -, \text{mix}$. Then, for every $\phi \in C_c^1(\mathbb{R}^2)$,

$$\begin{aligned} \int_{\mathbb{R}^2} f \cdot \nabla\phi \, \mathrm{d}x = &\int_{tc(\alpha)=0} (f_+^+ - f_-^-) \cdot \partial_\alpha z^\perp (\phi \circ z) \, \mathrm{d}\alpha \\ &+ \sum_{a=\pm} a \int_{tc(\alpha)>0} (f_a^a - f_a^{\text{mix}}) \cdot \partial_\alpha z_a^\perp (\phi \circ z_a) \, \mathrm{d}\alpha. \end{aligned}$$

Proof. First of all we split the integral over $\mathbb{R}^2$ into $\Omega_+(t)$, $\Omega_-(t)$ and $\Omega_{\text{mix}}(t)$. On the one hand, by applying the Gauss divergence theorem on the regular domains $\Omega_\pm(t)$, we get

$$\int_{\Omega_a(t)} f \cdot \nabla\phi \, \mathrm{d}x = a \int_{\mathbb{T}} f_a^a \cdot \partial_\alpha z_a^\perp (\phi \circ z_a) \, \mathrm{d}\alpha, \qquad a = \pm,$$

where we have applied that $\nabla \cdot f = 0$ outside $z_+(t) \cup z_-(t)$ and that the normal vector to $\partial\Omega_\pm(t)$ pointing outward is $\pm\partial_\alpha z_\pm(t)^\perp$. This concludes the proof for $t = 0$. Now we pay special attention to the cusp singularities in $\Omega_{\text{mix}}(t)$ for $0 < t \le T$. For any $\varepsilon > 0$ we define

$$\Omega_{\text{mix}}^\varepsilon(t) := \{z_\lambda(t, \alpha) \,:\, c(\alpha) > \varepsilon,\, \lambda \in (-1, 1)\},$$

which forms an exhaustion by Lipschitz domains of $\Omega_{\text{mix}}(t)$. Hence, we can apply first the dominated convergence and then the Gauss divergence theorem on $\Omega_{\text{mix}}^\varepsilon(t)$ to obtain

$$\int_{\Omega_{\text{mix}}(t)} f \cdot \nabla\phi \, \mathrm{d}x = \lim_{\varepsilon \to 0} \int_{\Omega_{\text{mix}}^\varepsilon(t)} f \cdot \nabla\phi \, \mathrm{d}x = \lim_{\varepsilon \to 0} \int_{\partial\Omega_{\text{mix}}^\varepsilon(t)} (f^{\text{tur}} \cdot n^\varepsilon)\phi \, \mathrm{d}\sigma,$$

where f^{tur} denotes the limit of f on $\partial\Omega_{\text{mix}}^\varepsilon(t)$ and n^ε is the unit normal vector to $\partial\Omega_{\text{mix}}^\varepsilon(t)$ pointing outward. Since $f \in L^\infty$ it follows that the contribution of the boundary integral on $c(\alpha) = \varepsilon$, $\lambda \in (-1, 1)$ is zero in the limit $\varepsilon \to 0$. Therefore, we deduce that

$$\int_{\Omega_{\text{mix}}(t)} f \cdot \nabla\phi \, \mathrm{d}x = -\sum_{a=\pm} a \int_{tc(\alpha)>0} f_a^{\text{mix}} \cdot \partial_\alpha z_a^\perp (\phi \circ z_a) \, \mathrm{d}\alpha.$$

This concludes the proof. ☐

6.2.1 The subsolution

The density

Following [70, 113] we declare $\bar{\rho} = 0$ on Ω_{mix}:

$$\bar{\rho}(t,x) := \mathbb{1}_{\Omega_+(t)}(x) - \mathbb{1}_{\Omega_-(t)}(x), \tag{6.41}$$

with $\Omega_\pm(t)$ given in (6.34). As a result, $\partial_1\bar{\rho}(t)$ is a Dirac measure supported on $\partial\Omega_+(t)\cup\partial\Omega_-(t)$ with density $(\partial_\alpha z_a(t))_2$ on each $\partial\Omega_a(t)$ for $a=\pm$.

Lemma 6.2.2. *For every $\phi \in C_c^1(\mathbb{R}^2)$ and $0 \le t \le T$,*

$$\int_{\mathbb{R}^2} \bar{\rho}(t)\partial_1\phi \,\mathrm{d}x = -\sum_{a=\pm}\int_{\mathbb{T}} (\partial_\alpha z_a(t,\alpha))_2 \phi(z_a(t,\alpha))\,\mathrm{d}\alpha.$$

Proof. It follows from Lemma 6.2.1 applied to $f = \bar{\rho}$ because, in this case, we have $1\cdot\partial_\alpha z_a^\perp = -(\partial_\alpha z_a)_2$, $\bar{\rho}_a^a = a$ and $\bar{\rho}_a^{\mathrm{mix}} = 0$. □

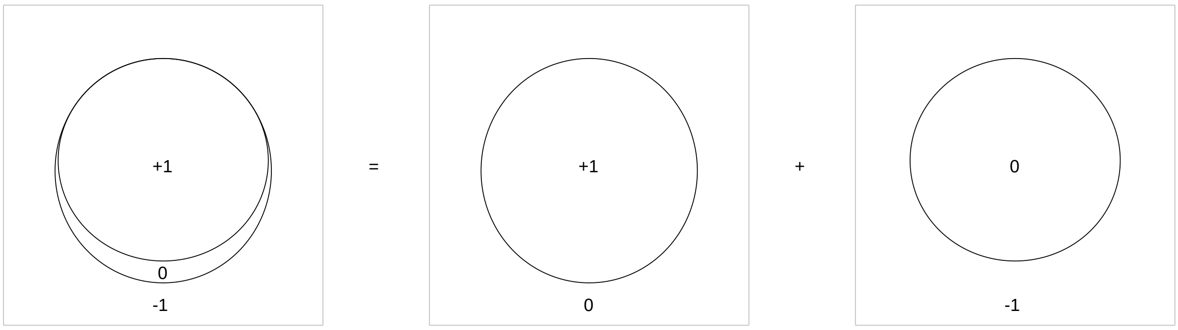


Figure 6.2: The macroscopic density $\bar{\rho}$ (6.41) can be decomposed as the sum of the contribution of ρ_+ on $\Omega_+\cup\Omega_{\mathrm{mix}}$ and ρ_- on $\Omega_-\cup\Omega_{\mathrm{mix}}$. In this way, the cusp singularities in Ω_{mix} can be understood as the superposition of regular domains.

The velocity

In view of Lemma 6.2.2 we define $\bar{v}$ by means of the Biot-Savart law (cf. (6.20))

$$\bar{v}(t,x)^* = -\frac{1}{2\pi i}\sum_{b=\pm}\int_{\mathbb{T}} \frac{(\partial_\alpha z_b(t,\beta))_2}{x - z_b(t,\beta)}\,\mathrm{d}\beta, \qquad x \neq z_b(t,\beta). \tag{6.42}$$

Observe that $\bar{v}$ is continuous (indeed $C_t^1 C^\omega$) outside $\partial\Omega_+\cup\partial\Omega_-$. Proposition 6.2.1 shows hat $\bar{v}$ satisfies the equations (6.31b)(6.31c), the boundedness condition (6.33) and that has well-defined continuous limits (6.40). Proposition 6.2.2 shows that the normal component of $\bar{v}$ on $\partial\Omega_+\cup\partial\Omega_-$ is well defined and continuous. Figure 6.2 explains why this is not surprising.

Proposition 6.2.1. *Let $\bar{\rho}$ be as in* (6.41)*. The unique velocity satisfying* (6.31b)(6.31c) *which additionally vanishes as $|x|\to\infty$ is precisely* (6.42)*. Moreover, $\bar{v}$ is uniformly bounded on $[0,T]\times\mathbb{R}^2$ and has well-defined continuous limits* (6.40)*.*

Proof. Step 1. $\bar{v}$ *in* (6.42) *is uniformly bounded and has well-defined continuous limits* (6.40). First of all notice that $\bar{v}$ is continuous (indeed $C_t^1 C^\omega$) outside $\partial\Omega_+ \cup \partial\Omega_-$. Moreover, for any $(t, x) \notin \partial\Omega_+ \cup \partial\Omega_-$ it holds that

$$|\bar{v}(t,x)| = \frac{1}{2\pi}\left|\sum_{b=\pm}\int_{\mathbb{T}}\left(\frac{1}{x - z_b(t,\beta)} - \frac{1}{x - z_b(t,0)}\right)(\partial_\alpha z_b(t,\beta))_2 \,\mathrm{d}\beta\right| \leq \frac{\ell_\circ^2}{8\pi}\sum_{b=\pm}\frac{\|\partial_\alpha z_b\|^2_{C_t C^0}}{\mathrm{dist}(x, \partial\Omega_b(t))^2},$$

that is, $\bar{v}$ decays as $|x|^{-2}$ when $|x| \to \infty$.

Let us manipulate the expression (6.42) to help better understand the behavior of $\bar{v}$ near the boundary $\partial\Omega_+ \cup \partial\Omega_-$. We will use de index w.r.t. $z_b(t)$ of points x outside $\partial\Omega_+(t) \cup \partial\Omega_-(t)$:

$$\mathrm{Ind}_{z_b(t)}(x) := \frac{1}{2\pi i}\int_{z_b(t)}\frac{\mathrm{d}z}{x - z} = \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{\partial_\alpha z_b(t,\beta)}{x - z_b(t,\beta)}\,\mathrm{d}\beta.$$

Recall that $z(t)$ is clockwise oriented ($\circlearrowright$). Hence, the Cauchy's argument principle yields

$$\begin{aligned} &\mathrm{Ind}_{z_+(t)}(x) = \mathbb{1}_{\Omega_+(t)}(x), \\ &\mathrm{Ind}_{z_-(t)}(x) = 1 - \mathbb{1}_{\Omega_-(t)}(x). \end{aligned} \tag{6.43}$$

In order to compute the limits (6.40), for any x outside $\partial\Omega_+(t) \cup \partial\Omega_-(t)$ but close enough and $a = \pm$, we take $\alpha(x, a) \in \mathbb{T}$ minimizing $|x - z_a(t,\alpha)|$. Then, a straightforward identity of complex numbers yields

$$\begin{aligned} &\bar{v}(t,x)^* \\ &= \sum_{b=\pm}\left(\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{(\partial_\alpha z_b(t,\beta))_2}{x - z_b(t,\beta)}\left(\frac{\partial_\alpha z_b(t,\beta)}{(\partial_\alpha z_b(t,\beta))_2}\frac{(\partial_\alpha z_b(t,\alpha))_2}{\partial_\alpha z_b(t,\alpha)} - 1\right)\mathrm{d}\beta - \frac{(\partial_\alpha z_b(t,\alpha))_2}{\partial_\alpha z_b(t,\alpha)}\mathrm{Ind}_{z_b(t)}(x)\right) \\ &= \sum_{b=\pm}\left(\frac{1}{2\pi i}\frac{1}{\partial_\alpha z_b(t,\alpha)}\int_{\mathbb{T}}\frac{\partial_\alpha z_b(t,\alpha)\cdot\partial_\alpha z_b(t,\beta)^\perp}{x - z_b(t,\beta)}\,\mathrm{d}\beta - \frac{(\partial_\alpha z_b(t,\alpha))_2}{\partial_\alpha z_b(t,\alpha)}\mathrm{Ind}_{z_b(t)}(x)\right). \end{aligned} \tag{6.44}$$

On the one hand, the regularity conditions (6.36)-(6.39) allow to apply the dominated convergence theorem on the first term in (6.44) as $x \to z_a(t,\alpha)$. On the other hand, the limits $\Omega_r(t) \ni x \to z_a(t,\alpha)$ in the second term in (6.44) change depending on the region $r = +, -, \mathrm{mix}$ where x is coming from due to (6.43). Therefore, $\bar{v}$ has well-defined continuous limits (6.40)

$$\begin{aligned} \bar{v}_+^+ &= V_+ - \frac{(\partial_\alpha z_-)_2}{(\partial_\alpha z_-)^*} - \frac{(\partial_\alpha z_+)_2}{(\partial_\alpha z_+)^*}, \\ \bar{v}_+^{\mathrm{mix}} &= V_+ - \frac{(\partial_\alpha z_-)_2}{(\partial_\alpha z_-)^*}, \\ \bar{v}_-^{\mathrm{mix}} &= V_- - \frac{(\partial_\alpha z_-)_2}{(\partial_\alpha z_-)^*}, \\ \bar{v}_-^- &= V_-, \end{aligned} \tag{6.45}$$

where

$$V_a := \sum_{b=\pm}V_{a,b} \quad \text{with} \quad V_{a,b}(t,\alpha)^* := \frac{1}{2\pi i}\frac{1}{\partial_\alpha z_b(t,\alpha)}\int_{\mathbb{T}}\frac{\partial_\alpha z_b(t,\alpha)\cdot\partial_\alpha z_b(t,\beta)^\perp}{z_a(t,\alpha) - z_b(t,\beta)}\,\mathrm{d}\beta,$$

for $(t,\alpha)\in[0,T]\times\mathbb{T}$ and $a,b=\pm$. Finally, it follows that $\bar{v}$ is uniformly bounded on $[0,T]\times\mathbb{R}^2$.

Step 2. $\bar{v}$ satisfies (6.31b)(6.31c). Observe that $\bar{v}(t)^*$ is holomorphic outside $\partial\Omega_+(t)\cup\partial\Omega_-(t)$ for all $0\leq t\leq T$. Thus, the Cauchy-Riemann equations imply that $\nabla\cdot\bar{v}(t)=\nabla^\perp\cdot\bar{v}(t)=0$ outside $\partial\Omega_+(t)\cup\partial\Omega_-(t)$. Notice also that $z_+=z_-$ and $V_+=V_-$ on $tc(\alpha)=0$. In particular,

$$\begin{aligned}
\bar{v}_+^+-\bar{v}_-^- &= -\left(\frac{(\partial_\alpha z_+)_2}{(\partial_\alpha z_+)^*}+\frac{(\partial_\alpha z_-)_2}{(\partial_\alpha z_-)^*}\right) \quad \text{on} \quad tc(\alpha)=0,\\
\bar{v}_a^a-\bar{v}_a^{\text{mix}} &= -a\frac{(\partial_\alpha z_a)_2}{(\partial_\alpha z_a)^*} \qquad\qquad\qquad \text{on} \quad tc(\alpha)>0.
\end{aligned}$$

Let $\phi\in C_c^1(\mathbb{R}^2)$ and $0<t\leq T$. Then, by applying Lemma 6.2.1 to $f=\bar{v}$ and $\bar{v}^\perp$, we deduce that

$$\begin{aligned}
\int_{\mathbb{R}^2}\bar{v}\cdot\nabla\phi\,\mathrm{d}x &= -\sum_{a=\pm}\int_{\mathbb{T}}\frac{(\partial_\alpha z_a)_2}{(\partial_\alpha z_a)^*}\cdot\partial_\alpha z_a^\perp(\phi\circ z_a)\,\mathrm{d}\alpha=0,\\
\int_{\mathbb{R}^2}\bar{v}\cdot\nabla^\perp\phi\,\mathrm{d}x &= \sum_{a=\pm}\int_{\mathbb{T}}\frac{(\partial_\alpha z_a)_2}{(\partial_\alpha z_a)^*}\cdot\partial_\alpha z_a(\phi\circ z_a)\,\mathrm{d}\alpha=\sum_{a=\pm}\int_{\mathbb{T}}(\partial_\alpha z_a)_2(\phi\circ z_a)\,\mathrm{d}\alpha.
\end{aligned}$$

These identities jointly with Lemma 6.2.2 imply that $\bar{v}$ satisfies (6.31b)(6.31c).

Step 3. Uniqueness. Finally, it is easy to check that any solution to (6.31b)(6.31c) has the form $\bar{u}=\bar{v}+f^*$ for some (time-dependent) entire function f. Thus, if $\bar{u}$ vanishes as $|x|\to\infty$ too, the Liouville's theorem implies that $f=0$. □

In the next lemma we deal with the normal component of $\bar{v}$ at the boundary of the mixing zone $\partial\Omega_+(t)\cup\partial\Omega_-(t)$. We will use the notation from Lemma 6.2.1 for the outer an inner limits and the operators B,B_a defined in the intro (6.14), (6.15).

Proposition 6.2.2. *Let $a=\pm$ and $r=+,-\text{mix}$. Then, it holds that*

$$(\bar{v}_a^r-B_a)\cdot\partial_\alpha z_a^\perp=0,$$

on $[0,T]\times\mathbb{T}$. In particular,

$$(\bar{v}_a^r-B)\cdot\partial_\alpha z^\perp=0 \quad \textit{on} \quad tc(\alpha)=0.$$

Proof. Let x be outside $\partial\Omega_+(t)\cup\partial\Omega_-(t)$ but close enough. Firstly, using that $(\mathrm{Ind}_{z_b(t)}(x))_2=0$, it follows that the velocity (6.42) can be written as (cf. (6.20)(6.21))

$$\bar{v}(t,x)=-\frac{1}{2\pi}\sum_{b=\pm}\int_{\mathbb{T}}\left(\frac{1}{x-z_b(t,\beta)}\right)_1\partial_\alpha z_b(t,\beta)\,\mathrm{d}\beta.$$

In particular,

$$\bar{v}(t,x)\cdot\partial_\alpha z_a(t,\alpha)^\perp=\frac{1}{2\pi}\sum_{b=\pm}\int_{\mathbb{T}}\left(\frac{1}{x-z_b(t,\beta)}\right)_1(\partial_\alpha z_a(t,\alpha)-\partial_\alpha z_b(t,\beta))\,\mathrm{d}\beta\cdot\partial_\alpha z_a(t,\alpha)^\perp,$$

where we take $\alpha\in\mathbb{T}$ as in (6.44). Thus, it remains to show that we can take the limit $x\to z_a(t,\alpha)$ in the r.h.s. above. By writing $z_a=z_b-i(a-b)tc\tau$, we split the above integrals into

$$\int_{\mathbb{T}}\left(\frac{1}{x-z_b(t,\beta)}\right)_1(\partial_\alpha z_b(t,\alpha)-\partial_\alpha z_b(t,\beta))\,\mathrm{d}\beta-i(a-b)t\partial_\alpha(c(\alpha)\tau(\alpha))\left(\int_{\mathbb{T}}\frac{\mathrm{d}\beta}{x-z_b(t,\beta)}\right)_1.$$

Analogously to (6.44), the regularity conditions (6.36)-(6.39) allow to apply the dominated convergence theorem on the first term as $x \to z_a(t,\alpha)$. Notice that the second term vanishes for $(a-b)tc(\alpha) = 0$. Otherwise, we can consider directly $x \to z_a(t,\alpha)$. This implies the first statement. As a by-product, we have seen that $B_{a,b}$ is split into

$$
\begin{aligned}
B_{a,b}(t,\alpha) = \frac{1}{2\pi}\int_{\mathbb{T}}\left(\frac{1}{z_a(t,\alpha)-z_b(t,\beta)}\right)_1 (\partial_\alpha z_b(t,\alpha) - \partial_\alpha z_b(t,\beta))\,\mathrm{d}\beta \\
- i(a-b)t\partial_\alpha(c(\alpha)\tau(\alpha))\frac{1}{2\pi}\left(\int_{\mathbb{T}}\frac{\mathrm{d}\beta}{z_a(t,\alpha)-z_b(t,\beta)}\right)_1.
\end{aligned}
\tag{6.46}
$$

Finally, the second statement follows from the fact that $z_+ = z_-$ and $B_+ = B_- = B$ on $tc(\alpha) = 0$. $\square$

The relaxed momentum

In view of the inequality (6.32), it seems suitable to define $\bar{m}$ as

$$
\bar{m} := \bar{\rho}\bar{v} - (1-\bar{\rho}^2)(\gamma + \tfrac{1}{2}i),
\tag{6.47}
$$

in terms of some $\gamma \in C(\Omega_{\mathrm{mix}};\mathbb{R}^2)$ to be determined ([126, 22, 70, 113]). Moreover, for any $0 < t \leq T$ we assume that $\gamma(t) \in C^1(\Omega_{\mathrm{mix}}(t))$ with continuous limits

$$
\gamma_a(t,\alpha) := \lim_{\Omega_{\mathrm{mix}}(t)\ni x \to z_a(t,\alpha)} \gamma(t,x),
$$

whenever $z_a(t,\alpha) \in \partial\Omega_{\mathrm{mix}}(t)$ for $a = \pm$ and $c(\alpha) > 0$.

6.2.2 Compatibility between the mixing zone and the subsolution

In the next proposition we derive the conditions for (c, z, γ) under which the corresponding $(\bar{\rho}, \bar{v}, \bar{m})$ given in (6.41),(6.42) and(6.47) becomes a subsolution.

Proposition 6.2.3. *Assume that (c,z) satisfies (6.36)-(6.39) for some $T > 0$. The triplet $(\bar{\rho}, \bar{v}, \bar{m})$ given in (6.41),(6.42) and (6.47) defines a subsolution to IPM if and only if the triplet (c, z, γ) satisfies the following equations on $\partial\Omega_a$ for $a = \pm$*

$$
(\partial_t z - B)\cdot \partial_\alpha z^\perp = 0 \quad \text{on} \quad tc(\alpha) = 0,
\tag{6.48a}
$$

$$
(\partial_t z_a - B_a - a(\gamma_a + \tfrac{1}{2}i))\cdot \partial_\alpha z_a^\perp = 0 \quad \text{on} \quad tc(\alpha) > 0,
\tag{6.48b}
$$

and the following conditions on Ω_{mix}

$$
\nabla\cdot\gamma = 0,
\tag{6.49a}
$$

$$
|\gamma| < \tfrac{1}{2}.
\tag{6.49b}
$$

Proof. Recall that $(\bar{\rho}, \bar{v}, \bar{m})$ already satisfies the equations (6.31b)(6.31c) and the conditions (6.32a) (6.33) (see Prop. 6.2.1). Moreover, it is clear that $(\bar{\rho}, \bar{v}, \bar{m})$ satisfies the equation (6.31a) on Ω_{mix} and the inequality (6.32b) if and only if (6.49) holds. Thus, it remains to analyze (6.31a) outside Ω_{mix}.

Let $\phi \in C_c^1(\mathbb{R}^3)$ and $0 < t \leq T$. On the one hand, since $\bar{\rho} = 0$ on Ω_{mix} and $\bar{\rho} = \pm 1$ on the regular domains $\Omega_{\pm}(t)$, an integration by parts yields

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}^2} \bar{\rho}\partial_t\phi \,\mathrm{d}x\,\mathrm{d}s - \int_{\mathbb{R}^2} \bar{\rho}(t)\phi(t)\,\mathrm{d}x + \int_{\mathbb{R}^2} \rho^\circ\phi^\circ\,\mathrm{d}x \\
&= -2\int_0^t \int_{c(\alpha)=0} \partial_t z \cdot \partial_\alpha z^\perp (\phi\circ Z)\,\mathrm{d}\alpha\,\mathrm{d}s - \sum_{a=\pm}\int_0^t\int_{c(\alpha)>0} \partial_t z_a \cdot \partial_\alpha z_a^\perp(\phi\circ Z_a)\,\mathrm{d}\alpha\,\mathrm{d}s,
\end{aligned}
$$

where $Z(t,\alpha) := (t, z(t,\alpha))$ and $Z_a(t,\alpha) := (t, z_a(t,\alpha))$. On the other hand, notice (6.47) reads as

$$
\bar{m}(t,x) = \begin{cases} \pm\bar{v}(t,x), & x \in \Omega_{\pm}(t), \\ -(\gamma(t,x) + \frac{1}{2}i), & x \in \Omega_{\mathrm{mix}}(t). \end{cases}
$$

In particular, $\bar{m}$ satisfies the assumptions in Lemma 6.2.1. Therefore, it follows that

$$
\begin{aligned}
\int_{\mathbb{R}^2} \bar{m}\cdot\nabla\phi\,\mathrm{d}x &= 2\int_{c(\alpha)=0} B\cdot\partial_\alpha z^\perp(\phi\circ Z)\,\mathrm{d}\alpha \\
&+ \sum_{a=\pm}\int_{c(\alpha)>0} B_a\cdot\partial_\alpha z_a^\perp(\phi\circ Z_a)\,\mathrm{d}\alpha \\
&+ \sum_{a=\pm} a\int_{c(\alpha)>0} (\gamma_a + \tfrac{1}{2}i)\cdot\partial_\alpha z_a^\perp(\phi\circ Z_a)\,\mathrm{d}\alpha,
\end{aligned}
$$

where we have applied Proposition 6.2.2 in the first two lines above. In summary, we have seen that

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}^2} (\bar{\rho}\partial_t\phi + \bar{m}\cdot\nabla\phi)\,\mathrm{d}x\,\mathrm{d}s - \int_{\mathbb{R}^2}\bar{\rho}(t)\phi(t)\,\mathrm{d}x + \int_{\mathbb{R}^2}\rho^\circ\phi^\circ\,\mathrm{d}x \\
&= -2\int_0^t\int_{c(\alpha)=0} (\partial_t z - B)\cdot\partial_\alpha z^\perp(\phi\circ Z)\,\mathrm{d}\alpha\,\mathrm{d}s \\
&- \sum_{a=\pm}\int_0^t\int_{c(\alpha)>0} ((\partial_t z_a - B_a - a(\gamma_a + \tfrac{1}{2}i))\cdot\partial_\alpha z_a^\perp)(\phi\circ Z_a)\,\mathrm{d}\alpha\,\mathrm{d}s.
\end{aligned}
$$

This concludes the proof. □

We conclude this section by showing that we can construct $\gamma(t,x)$ satisfying the requirements in Proposition 6.2.3 provided that (c,z) satisfies certain conditions. Observe that $\{c(\alpha) > 0\}$ is open and thus a (countable) union of disjoint intervals (α_1, α_2). Recall the definition of B and D from (6.14) and (6.26).

Lemma 6.2.3. *Assume that (c,z) satisfies (6.36)-(6.39) for some $T > 0$. Assume further that the following conditions hold uniformly on $\{c(\alpha) > 0\}$*

$$
|2c(\alpha) + \partial_\alpha z_1^\circ(\alpha)| < 1, \tag{6.50}
$$

and

$$
\partial_t z - B_a = o(1), \tag{6.51a}
$$

$$
\frac{1}{tc(\alpha)}\int_{\alpha_1}^{\alpha} ((\partial_t z - B)\cdot\partial_\alpha z^\perp + tD\cdot\partial_\alpha(c\tau))\,\mathrm{d}\alpha' = o(1), \tag{6.51b}
$$

as $t \to 0$, for $a = \pm$ and $\alpha \in (\alpha_1, \alpha_2)$ connected component of $\{c(\alpha) > 0\}$. Then, there exists $0 < T' \leq T$ and $\gamma(t,\alpha)$ satisfying (6.48b)-(6.49) *as long as $0 < t \leq T'$.*

Proof. Step 1. Analysis of (6.48b)-(6.49). For simplicity we may assume w.l.o.g. that there is one connected component $(\alpha_1, \alpha_2) = \{c(\alpha) > 0\}$. Recall that $(\alpha, \lambda) \mapsto z_\lambda(t,\alpha)$ is a diffeomorphism from $(\alpha_1, \alpha_2) \times (-1, 1)$ to $\Omega_{\mathrm{mix}}(t)$ (cf. Rem. 6.2.3). In particular, since $\Omega_{\mathrm{mix}}(t)$ is simply-connected, (6.49a) implies that $\gamma(t) = \nabla^\perp g(t)$ for some $g(t) \in C^1(\Omega_{\mathrm{mix}}(t))$ to be determined. Moreover, g can be defined in terms of some G in (α, λ)-coordinates as

$$g(Z(t,\alpha,\lambda)) := G(t,\alpha,\lambda),$$

where $Z(t,\alpha,\lambda) := (t, z_\lambda(t,\alpha))$. Notice that (recall $z_\lambda = z - \lambda t c \tau^\perp$)

$$\partial_\alpha G = (\nabla g \circ Z) \cdot \partial_\alpha z_\lambda, \qquad \partial_\lambda G = -tc(\nabla g \circ Z) \cdot \tau^\perp.$$

On the one hand, the boundary conditions (6.48b) for γ read as

$$\partial_\alpha G(t,\alpha,a) = (a(\partial_t z - B_a) - i(c\tau + \tfrac{1}{2})) \cdot \partial_\alpha z_a^\perp, \qquad a = \pm. \tag{6.52}$$

On the other hand, for (6.49b) notice that

$$\nabla g \circ Z = \frac{1}{\partial_\alpha z_\lambda \cdot \tau} \left(\partial_\alpha G \tau - \frac{1}{tc} \partial_\lambda G \partial_\alpha z_\lambda^\perp \right). \tag{6.53}$$

In particular,

$$\partial_\alpha z_\lambda \cdot \tau = \partial_\alpha z^\circ \cdot \tau + t \left(\fint_0^t \partial_t z \,\mathrm{d}s - \lambda \partial_\alpha (c\tau^\perp) \right) \cdot \tau = \partial_\alpha z^\circ \cdot \tau + O(t),$$

with $\partial_\alpha z^\circ \cdot \tau > 0$ uniformly on $c(\alpha) > 0$ by (6.38). Therefore, assuming that c satisfies (6.50), it is enough to find G satisfying (6.52) and the following growth conditions

$$\partial_\alpha G = o(1) - (c\tau + \tfrac{1}{2}) \cdot \partial_\alpha z_\lambda, \qquad \frac{1}{tc} \partial_\lambda G = o(1), \tag{6.54}$$

uniformly on $c(\alpha) > 0$ as $t \to 0$, because in this case (recall $\tau = \partial_\alpha z^\circ$ with $|\partial_\alpha z^\circ| = 1$)

$$|\gamma| = |\nabla g| \leq \left| c + \tfrac{1}{2} \tfrac{\partial_\alpha z_1^\circ}{\partial_\alpha z^\circ \cdot \tau} \right| + o(1) < \tfrac{1}{2}. \tag{6.55}$$

Step 2. Ansatz for G. We declare

$$G(t,\alpha,\lambda) := \int_{\alpha_1}^{\alpha} \left(\sum_{a=\pm} \frac{\lambda + a}{2} (\partial_t z - B_a) \cdot \partial_\alpha z_a^\perp - (c\tau + \tfrac{1}{2}) \cdot \partial_\alpha z_\lambda \right) \mathrm{d}\alpha'. \tag{6.56}$$

Hence, it follows that

$$\partial_\alpha G = \sum_{a=\pm} \frac{\lambda + a}{2} (\partial_t z - B_a) \cdot \partial_\alpha z_a^\perp - (c\tau + \tfrac{1}{2}) \cdot \partial_\alpha z_\lambda,$$
$$\partial_\lambda G = \int_{\alpha_1}^{\alpha} \Big((\partial_t z - B) \cdot \partial_\alpha z^\perp + tD \cdot \partial_\alpha (c\tau) \Big) \,\mathrm{d}\alpha'.$$

Notice that (6.52) is satisfied. Finally, assuming that z satisfies (6.51), then (6.54) holds. □

Remark 6.2.4. Following the previous proof, notice that for $z(t,\alpha) = (\alpha, f(t,\alpha))$ and $\tau = (-1,0)$ as in [70, 113], the admissible regime for $c(\alpha)$ reads as

$$|2c(\alpha) - 1| < 1,$$

which clear is incompatibly with $c(\alpha) = 0$. Remarkably, the authors in [113] achieved that $c(\alpha) \to 0$ in the limiting case $|\alpha| \to \infty$ for $\alpha \in \mathbb{R}$.

In view of Proposition 6.2.3 and Lemma 6.2.3, we need to find a growth-rate $c(\alpha)$ with certain regularity (6.36) and satisfying the inequality (6.50), and a time-dependent pseudo-interface $z(t,\alpha)$ satisfying the regularity assumptions (6.37)-(6.39) and the relations (6.48a)(6.51).

6.3 The growth-rate

In this section we declare a suitable growth-rate c, and also a partition of the unity $\{\psi_0, \psi_1\}$ as discussed in the intro, in terms of $z^\circ \in C^{1,\delta}(\mathbb{T};\mathbb{R}^2)$. Let us recall what we need. On the one hand, we must construct a regular enough c (6.36) satisfying the inequality (6.50) uniformly on $\{c(\alpha) > 0\}$. At the same time, for the relation (6.51b) it is convenient to control the monotonicity of c near the boundary of $\{c(\alpha) > 0\}$. On the other hand, we shall construct $\{\psi_0, \psi_1\}$ subordinated to $\{\partial_\alpha z_1^\circ(\alpha) > 0\}$ and $\{c(\alpha) > 0\}$ respectively, and satisfying $\partial_\alpha z_1^\circ > 0$ uniformly on $\operatorname{supp}\psi_0$.

In view of Figure 6.3, it seems clear that we have enough flexibility to construct such functions. The aim of this section is to make it quantitatively.

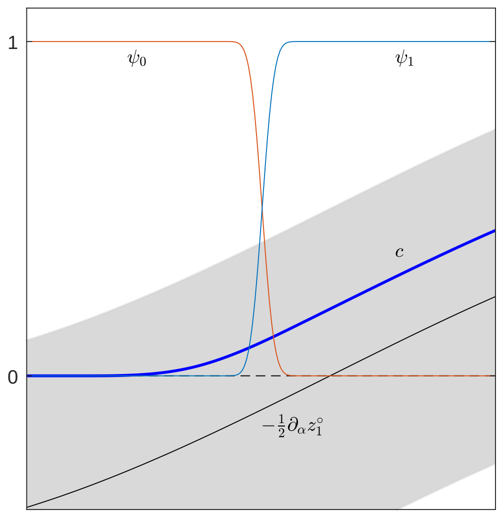


Figure 6.3: A regular growth-rate $c(\alpha) \geq 0$ (blue) satisfying the inequality $|2c(\alpha) + \partial_\alpha z_1^\circ(\alpha)| < 1$ (gray zone) uniformly on $\{c(\alpha) > 0\}$. A partition of the unity $\{\psi_0, \psi_1\}$ subordinated to $\{\partial_\alpha z_1^\circ(\alpha) > 0\}$ and $\{c(\alpha) > 0\}$ respectively, and satisfying $\partial_\alpha z_1^\circ > 0$ uniformly on $\operatorname{supp}\psi_0$.

Let us introduce the following sets I_η which will be very useful for these purposes.

Lemma 6.3.1. *Given* $-1 \leq \eta \leq 1$ *we denote*

$$I_\eta := \{\alpha \in \mathbb{T} \, : \, \partial_\alpha z_1^\circ(\alpha) < \eta\}. \tag{6.57}$$

This forms an ascending chain of open subsets of $\mathbb{T}$*. Furthermore, for any* $-1 < \eta_1 < \eta_2 < 1$*, we have* $I_{\eta_1} \subset\subset I_{\eta_2}$ *with*

$$\mathrm{dist}(\partial I_{\eta_1}, \partial I_{\eta_2}) \geq \left(\frac{\eta_2 - \eta_1}{|\partial_\alpha z_1^\circ|_{C^\delta}}\right)^{1/\delta}.$$

Proof. First of all notice that $\partial_\alpha z_1^\circ(\mathbb{T}) = [-1, 1]$. Hence, there is $\alpha_j \in \partial I_{\eta_j}$ for $j = 1, 2$, and thus

$$\eta_2 - \eta_1 = \partial_\alpha z_1^\circ(\alpha_2) - \partial_\alpha z_1^\circ(\alpha_1) \leq |\partial_\alpha z_1^\circ|_{C^\delta} \mathrm{dist}(\partial I_{\eta_1}, \partial I_{\eta_2})^\delta,$$

as we wanted to prove. $\square$

In order to interpolate between the Classical and the Mixing Muskat problem, we take a partition of the unity $\{\psi_0, \psi_1\} \subset C_c^\infty(\mathbb{T}; [0, 1])$ as follows. Firstly, we define the indicator function

$$\chi_{\eta,s}(\alpha) := \begin{cases} 1, & \mathrm{dist}(\alpha, \mathbb{T} \setminus I_\eta) > r, \\ 0, & \text{otherwise}, \end{cases} \qquad \text{with} \quad r := s \left(\frac{\eta}{|\partial_\alpha z_1^\circ|_{C^\delta}}\right)^{1/\delta}, \tag{6.58}$$

in terms of some parameters $0 < \eta, s < 1$ to be determined. With $\chi_{\eta,s}$ we declare

$$\psi_1 := \phi_r * \chi_{\eta,s}, \qquad \psi_0 := 1 - \psi_1, \tag{6.59}$$

where ϕ_r is a standard mollifier, namely $\phi_r(\alpha) := \frac{1}{r}\phi(\frac{\alpha}{r})$ for some fixed $\phi \in C_c^\infty(-1, 1)$ satisfying $\phi \geq 0$ and $\int \phi = 1$, and $r > 0$ is given in (6.58). Hence,

$$\|\partial_\alpha^k \psi_j\|_{L^\infty} \leq \|\partial_\alpha^k \phi\|_{L^1} r^{-k}, \qquad k \geq 0, \; j = 0, 1.$$

Among all the possible choices for c, we declare

$$c := \tfrac{1}{2}\phi_r * (\eta\chi_{\eta,s} + (\partial_\alpha z_1^\circ)_-), \tag{6.60}$$

where $(\partial_\alpha z_1^\circ)_- := -\min(\partial_\alpha z_1^\circ, 0)$. This c is smooth with

$$\|\partial_\alpha^k c\|_{L^\infty} \leq \tfrac{1}{2}(\eta + 1)\|\partial_\alpha^k \phi\|_{L^1} r^{-k}, \qquad k \geq 0,$$

and satisfies $c \geq \psi_1 \eta/2$ with $\mathrm{supp}\, c = \mathrm{supp}\, \psi_1 \subset \bar{I}_\eta$. In the next lemma we show that we can take η and s in such a way that the inequality (6.50) holds.

Lemma 6.3.2. *The growth-rate* (6.60) *satisfies*

$$|2c + \partial_\alpha z_1^\circ| \leq \eta(2 + s^\delta) \quad \textit{on} \quad c(\alpha) > 0.$$

Proof. By writing $f = f_+ - f_-$ for $f = \partial_\alpha z_1^\circ$, we split

$$2c + \partial_\alpha z_1^\circ = \eta\psi_1 + f_+ + (\phi_r * f_- - f_-).$$

On the one hand, $\eta\psi_1 \leq \eta$ and also $f_+ \leq \eta$ on $\bar{I}_\eta$ ($\supset \mathrm{supp}\, c$) by (6.57). On the other hand, by (6.58)

$$|(\phi_r * f_- - f_-)(\alpha)| = \left|\int_{-r}^{r} (f_-(\alpha - \beta) - f_-(\alpha))\phi_r(\beta)\, \mathrm{d}\beta\right| \leq |f_-|_{C^\delta} r^\delta \leq s^\delta \eta.$$

This concludes the proof. $\square$

Let us assume from now on that $s < 1/3$. In the next lemma we show that $\partial_\alpha z_1^\circ > 0$ uniformly on $\operatorname{supp}\psi_0$ ($\supset\supset \mathbb{T}\setminus\operatorname{supp} c$), which will be crucial for the energy estimates in section 6.5.

Lemma 6.3.3. *It holds that*

$$\partial_\alpha z_1^\circ \geq \eta(1-(2s)^\delta) \quad on \quad \operatorname{supp}\psi_0.$$

Proof. Let $\alpha \in \operatorname{supp}\psi_0$. If $\alpha \notin I_\eta$, simply $\partial_\alpha z_1^\circ(\alpha) \geq \eta$ by (6.57). Assume now that $\alpha \in I_\eta$. Since $\psi_1 \equiv 1$ on the open set $I_\eta \setminus \bar{B}_{2r}(\partial I_\eta)$ by (6.58)(6.59), necessarily $\alpha \in \bar{B}_{2r}(\partial I_\eta)$. Hence, there is $\alpha_\eta \in \partial I_\eta$ satisfying $|\alpha - \alpha_\eta| \leq 2r$, and thus

$$\partial_\alpha z_1^\circ(\alpha) = \underbrace{\partial_\alpha z_1^\circ(\alpha_\eta)}_{=\eta} + \underbrace{\partial_\alpha z_1^\circ(\alpha) - \partial_\alpha z_1^\circ(\alpha_\eta)}_{\geq -|\partial_\alpha z_1^\circ|_{C^\delta}(2r)^\delta} \geq \eta(1-(2s)^\delta),$$

where we have applied (6.58). □

Next we turn to the behavior of c inside $\{c(\alpha) > 0\}$. Of course c is monotone in a neighborhood of $c(\alpha) = 0$ and away from zero outside it, but we give bounds for these properties that only depends on s, η, δ.

Lemma 6.3.4. *Let $I = (\alpha_1, \alpha_2)$ be a connected component of $\{c(\alpha) > 0\}$ and denote $\bar{\alpha} := \frac{1}{2}(\alpha_1 + \alpha_2)$. If $|I| < 2r(1/s - 1)$, then c is monotone on $[\alpha_1, \bar{\alpha}]$ and $[\bar{\alpha}, \alpha_2]$. Otherwise, c is monotone on each connected component of $I \cap B_{2r}(\partial I)$, while $c \geq \eta/2$ on $I \setminus B_{2r}(\partial I)$. Moreover $\psi_1\eta/2 < c$ everywhere.*

Proof. First of all observe that $\phi_r * (\partial_\alpha z_1^\circ)_- = 0$ (and so $c = \psi_1\eta/2$) outside $B_r(I_0)$.

Case $|I| < 2r(1/s - 1)$. Given $\alpha \in B_r(I_0)$, Lemma 6.3.1 and (6.58) imply that

$$\operatorname{dist}(\alpha, \partial I) \geq \operatorname{dist}(I_0, \partial I) - r \geq r(1/s - 1).$$

Thus, necessarily $\alpha \notin I$. Then, $I \cap B_r(I_0) = \emptyset$ and so $c = \psi_1\eta/2$ with ψ_1 monotone on $[\alpha_1, \bar{\alpha}]$ and $[\bar{\alpha}, \alpha_2]$ by construction (6.58)(6.59).

Case $|I| \geq 2r(1/s - 1)$. Given $\alpha \in B_{2r}(I_0)$ and $\beta \in B_r(\partial I)$, Lemma 6.3.1 and (6.58) imply that

$$|\alpha - \beta| \geq \operatorname{dist}(I_0, \partial I) - 3r \geq r(1/s - 3) > 0.$$

Hence, $B_{2r}(\partial I) \cap B_r(I_0) = \emptyset$ and so $c = \psi_1\eta/2$ on $I \cap B_{2r}(\partial I)$ with ψ_1 monotone on each connected component of $I \cap B_{2r}(\partial I)$. Finally, $c \geq \psi_1\eta/2$ with $\psi_1 \equiv 1$ on $I \setminus B_{2r}(\partial I)$. □

Finally, Lemma 6.3.4 implies the following estimates which will be useful to control (6.51b).

Corollary 6.3.1. *In the context of Lemma 6.3.4, let us denote*

$$I(\alpha) := \begin{cases} [\alpha_1, \alpha], & \alpha_1 \leq \alpha \leq \bar{\alpha}, \\ [\alpha, \alpha_2], & \bar{\alpha} < \alpha \leq \alpha_2. \end{cases}$$

Then, for $k = 0, 1$, we have

$$\int_{I(\alpha)} |\partial_\alpha^k c(\alpha')| \, \mathrm{d}\alpha' \lesssim c(\alpha),$$

in terms of $(\eta/|\partial_\alpha z_1^\circ|_{C^\delta})^{1/\delta}$, $1/\eta$ and $\|\partial_\alpha^k c\|_{L^1}$.

Proof. *Case* $|I| < 2r(1/s-1)$. For all $\alpha \in I$, the monotonicity of c on $I(\alpha)$ implies

$$\int_{I(\alpha)} |\partial_\alpha^k c(\alpha')| \,\mathrm{d}\alpha' \le |I(\alpha)|^{1-k} c(\alpha),$$

with $|I(\alpha)| \le r(1/s-1) = (1-s)(\eta/|\partial_\alpha z_1^\circ|_{C^\delta})^{1/\delta} \le (\eta/|\partial_\alpha z_1^\circ|_{C^\delta})^{1/\delta}$.

Case $|I| \ge 2r(1/s-1)$. For all $\alpha \in I \cap B_{2r}(\partial I)$, the previous argument works with $|I(\alpha)| \le 2r \le (\eta/|\partial_\alpha z_1^\circ|_{C^\delta})^{1/\delta}$. Finally, for all $\alpha \in I \setminus B_{2r}(\partial I)$, simply

$$\int_{I(\alpha)} |\partial_\alpha^k c(\alpha')| \,\mathrm{d}\alpha' \le \|\partial_\alpha^k c\|_{L^1} \frac{c(\alpha)}{\eta/2},$$

because $c(\alpha) \ge \eta/2$. □

From now on we fix the parameters η and s satisfying $\eta(2+s^\delta) < 1$ with $s < 1/3$. We remark that they are not necessarily very small (e.g. we can take $\eta = s = \frac{1}{4}$ for $\delta = 1$).

6.4 The pseudo-interface

We define our pseudo-interface z as the solution of the integro-differential equation

$$\begin{aligned} \partial_t z &= F(t, z^\circ, z), \\ z|_{t=0} &= z^\circ, \end{aligned} \tag{6.61}$$

given by the operator

$$F := \psi_0 E + \psi_1 E^{1)} - (t\kappa + i(tD^{(0)} \cdot \partial_\alpha(c\tau) + h\psi_1)\partial_\alpha z^\circ),$$

where τ is given in (6.12), $\{\psi_0, \psi_1\}$ is the partition of the unity we fixed in (6.59) and c the growth-rate (6.60). The operator $E(t, z^\circ, z)$ extending B outside $tc(\alpha) = 0$ was already introduced in (6.27),

$$E := \sum_{b=\pm} B_{b,b}.$$

Thus, it expands as

$$E(t,\alpha) := \frac{1}{2\pi} \sum_{b=\pm} \int_{\mathbb{T}} \left(\frac{1}{z_b(t,\alpha) - z_b(t,\beta)} \right)_1 (\partial_\alpha z_b(t,\alpha) - \partial_\alpha z_b(t,\beta)) \,\mathrm{d}\beta.$$

The term $E^{1)}(t, z^\circ)$ is

$$E^{1)} := E^{(0)} + tE^{(1)},$$

where $E^{(0)}(z^\circ)$ and $E^{(1)}(z^\circ)$ are

$$\begin{aligned} E^{(0)}(\alpha) &:= \frac{1}{2\pi} \sum_{b=\pm} \int_{\mathbb{T}} \left(\frac{1}{z^\circ(\alpha) - z^\circ(\beta)} \right)_1 (\partial_\alpha z^\circ(\alpha) - \partial_\alpha z^\circ(\beta)) \,\mathrm{d}\beta, \\ E^{(1)}(\alpha) &:= \frac{1}{2\pi} \sum_{b=\pm} \int_{\mathbb{T}} \left(\frac{1}{z^\circ(\alpha) - z^\circ(\beta)} \right)_1 (\partial_\alpha z_b^{(1)}(\alpha) - \partial_\alpha z_b^{(1)}(\beta)) \,\mathrm{d}\beta \\ &\quad - \frac{1}{2\pi} \sum_{b=\pm} \int_{\mathbb{T}} \left(\frac{z_b^{(1)}(\alpha) - z_b^{(1)}(\beta)}{(z^\circ(\alpha) - z^\circ(\beta))^2} \right)_1 (\partial_\alpha z^\circ(\alpha) - \partial_\alpha z^\circ(\beta)) \,\mathrm{d}\beta, \end{aligned} \tag{6.62}$$

with
$$z_b^{(1)} := E^{(0)} - bc\tau^{\perp}.$$
The terms $\kappa(z^\circ)$ and $D^{(0)}(z^\circ)$ are

$$\begin{aligned}\kappa(\alpha) &:= 2\left(\partial_\alpha^2 z^\circ \left(\frac{c\tau}{(\partial_\alpha z^\circ)^2}\right)_1 + i\left(\frac{1}{\partial_\alpha z^\circ}\right)_2 \partial_\alpha(c\tau)\right), \\ D^{(0)}(\alpha) &:= -i(c\tau + \tfrac{1}{2}).\end{aligned} \tag{6.63}$$

The term $h(t, z^\circ, z)$ is the time-dependent average defined on each connected component (α_1, α_2) of $\{c(\alpha) > 0\}$ as

$$h(t) := \frac{\displaystyle\int_{\alpha_1}^{\alpha_2} H \,\mathrm{d}\alpha}{\displaystyle\int_{\alpha_1}^{\alpha_2} \psi_1 \partial_\alpha z \cdot \partial_\alpha z^\circ \,\mathrm{d}\alpha}, \tag{6.64}$$

where

$$H(t,\alpha) := (E - B - t\kappa)\cdot\partial_\alpha z^\perp + \psi_1(E^{1)} - E)\cdot\partial_\alpha z^\perp + t(D - D^{(0)}\partial_\alpha z\cdot\partial_\alpha z^\circ)\cdot\partial_\alpha(c\tau).$$

We notice that, although h is piecewise constant, $h\psi_1$ is smooth in α (recall $\psi_1 \lesssim c$ by Lemma 6.3.1).

As we will see in the next lemmas, $E^{(0)} = E|_{t=0}$, $E^{(1)} = \partial_t E|_{t=0}$, $z_b^{(1)} = \partial_t z_b|_{t=0}$ and $D^{(0)} = D|_{t=0}$. Thus, $E^{1)}$ equals the first order expansion in time of E. In addition, we will see that $\kappa = \partial_t(E - B)|_{t=0}$.

In the rest of this section we will assume that there exists a solution z of equation (6.61) satisfying (6.37)-(6.39) and show that this implies that z satisfies (6.48a)(6.51) as well.

Theorem 6.4.1. *Let $z^\circ \in H^{k_\circ}(\mathbb{T};\mathbb{R}^2)$ be a closed chord-arc curve with $k_\circ \geq 6$. Assuming that, for some $T > 0$, there exists $z \in C_t H^{k_\circ - 2}$ with $\partial_t z \in C_t H^{k_\circ - 3}$ solving* (6.61) *and satisfying* (6.38)(6.39)*, then z satisfies* (6.48a)(6.51).

The proof of this theorem relies on the forthcoming Lemmas 6.4.1-6.4.5. We start by rewriting some of the terms suitably. Let us observe that, assuming (6.61), the l.h.s. of (6.51a) reads, for $a = \pm$, as

$$\partial_t z - B_a = (E - B_a) + \psi_1(E^{1)} - E) - (t\kappa + i(tD^{(0)}\cdot\partial_\alpha(c\tau) + h\psi_1)\partial_\alpha z^\circ), \tag{6.65}$$

and for (6.51b) we have

(6.66)

$$\begin{aligned}&(\partial_t z - B)\cdot\partial_\alpha z^\perp + tD\cdot\partial_\alpha(c\tau) \\ &= (E - B - t\kappa)\cdot\partial_\alpha z^\perp + \psi_1(E^{1)} - E)\cdot\partial_\alpha z^\perp + t(D - D^{(0)}\partial_\alpha z\cdot\partial_\alpha z^\circ)\cdot\partial_\alpha(c\tau) - h\psi_1\partial_\alpha z\cdot\partial_\alpha z^\circ.\end{aligned}$$

The core of the section is to prove that E remains close to B in L^∞. In Lemma 6.4.2 we show that $E - B_a = O(t(c + |\partial_\alpha c|))$. This is sufficient for (6.51a). Indeed, we show in Lemma 6.4.3 that $E - B - t\kappa = O(t^2(c + |\partial_\alpha c|))$, which is sufficient for (6.51b). In particular, this implies that $\kappa = \partial_t(E - B)|_{t=0}$ (a similar term appears in [70, 113]). Both estimates are based on a nice technical consequence of the argument principle Lemma 6.4.1 (a related argument appears in [103, Lemma 4.2]). In Lemma 6.4.4 we show that $D - D^{(0)}\partial_\alpha z \cdot \partial_\alpha z^\circ = O(t)$. The term h has been introduced because (6.51b) reads as

$$\int_{\alpha_1}^{\alpha} (H - h\psi_1 \partial_\alpha z \cdot \partial_\alpha z^\circ)\, \mathrm{d}\alpha' = c(\alpha) o(t).$$

This requires to obtain a cancellation for $\alpha = \alpha_2$, which is equivalent to (6.64). Finally, we show in Lemma 6.4.5 that h, $E - E^{1)} = O(t^2)$. All the ingredients are ready to prove Theorem 6.4.1.

In the proof of Lemmas 6.4.1-6.4.5 all the assumptions in Theorem 6.5.1 are valid. Let us recall the auxiliary Lemma 2.2.1.

Lemma 6.4.1. *For all $k \in \mathbb{N}$, the function*

$$C^k_{\lambda,\mu}(t,\alpha) := \int_{\mathbb{T}} \frac{\beta^{k-1}}{(z_\lambda(t,\alpha) - z_\mu(t,\alpha-\beta))^k}\, \mathrm{d}\beta, \tag{6.67}$$

is uniformly bounded on $c(\alpha) > 0$, $0 < t \leq T$ and $\lambda, \mu \in [-1,1]$ with $\lambda \neq \mu$.

Lemma 6.4.2. *For $0 \leq t \leq T$ it holds that*

$$E - B_a = O(t(c + |\partial_\alpha c|)).$$

Proof. Notice that $E = B = B_+ = B_-$ for $tc(\alpha) = 0$. Consider now $tc(\alpha) \neq 0$. Recall from (6.46) that $B_{a,b}$ is split into

$$B_{a,b} = A_{a,b} - i(a-b)t\partial_\alpha(c\tau)\frac{1}{2\pi}(C^1_{a,b})_1,$$

where $C^1_{a,b}$ is given in (6.67) and

$$A_{a,b} := \frac{1}{2\pi}\int_{\mathbb{T}} \left(\frac{1}{z_a - z_b'}\right)_1 (\partial_\alpha z_b - \partial_\alpha z_b')\, \mathrm{d}\beta.$$

Then, for $b = -a$, we have

$$E - B_a = B_{b,b} - B_{a,b} = B_{b,b} - A_{a,b} + iat\partial_\alpha(c\tau)\frac{1}{\pi}(C^1_{a,b})_1. \tag{6.68}$$

The last term is $O(t|\partial_\alpha(c\tau)|)$ by Lemma 6.4.1. For the first term, the fundamental theorem of calculus yields

$$\begin{aligned} B_{b,b} - A_{a,b} &= \frac{1}{2\pi}\int_{\mathbb{T}} \left(\frac{1}{z_b - z_b'} - \frac{1}{z_a - z_b'}\right)_1 (\partial_\alpha z_b - \partial_\alpha z_b')\, \mathrm{d}\beta \\ &= -\frac{a}{2\pi} tc \int_{\mathbb{T}}\int_{-1}^{1} \left(\frac{\tau^\perp}{(z_\lambda - z_b')^2}\right)_1 (\partial_\alpha z_b - \partial_\alpha z_b')\, \mathrm{d}\lambda\, \mathrm{d}\beta. \end{aligned} \tag{6.69}$$

Let us check that we can apply the Fubini's theorem. By using the regularity conditions (6.36)-(6.39), then (6.69) can be bounded by

$$tc\int_0^{\ell_\circ/2}\int_0^1 \frac{\beta}{\beta^2 + ((1-\lambda)tc)^2}\, \mathrm{d}\lambda\, \mathrm{d}\beta = tc\int_0^1 \int_0^{\frac{\ell_\circ/2}{(1-\lambda)tc}} \frac{\beta}{1+\beta^2}\, \mathrm{d}\beta\, \mathrm{d}\lambda < \infty. \tag{6.70}$$

Hence, the Fubini's theorem allows to interchange the order of integration of λ and β in (6.69). Then, by adding and subtracting $\beta\partial_\alpha^2 z_b$ in (6.69), we get

$$\begin{aligned} B_{b,b} - A_{a,b} = &- \frac{a}{2\pi} tc \int_{-1}^{1} \int_{\mathbb{T}} \left(\frac{\tau^\perp}{(z_\lambda - z_b')^2} \right)_1 (\partial_\alpha z_b - \partial_\alpha z_b' - \beta \partial_\alpha^2 z_b) \, \mathrm{d}\beta \, \mathrm{d}\lambda \\ &- \frac{a}{2\pi} tc \partial_\alpha^2 z_b \left(\tau^\perp \int_{-1}^{1} C_{\lambda,b}^2 \, \mathrm{d}\lambda \right)_1, \end{aligned} \tag{6.71}$$

where, by applying the Taylor's theorem on the first term and Lemma 6.4.1 on the second one, we see that (6.71) is $O(tc)$ in terms of $\mathcal{C}(c,z)$ and $\|\partial_\alpha^2 z_b\|_{C_t C^{0,\delta}}$. □

The next lemma shows that indeed $\kappa = \partial_t (E - B)|_{t=0}$.

Lemma 6.4.3. *For $0 \le t \le T$ it holds that*

$$E - B - t\kappa = O(t^2(c + |\partial_\alpha c|)).$$

Proof. Notice that

$$E - B = \tfrac{1}{2}(E - B_-) + \tfrac{1}{2}(E - B_+),$$

with $E - B_a$ given in (6.68). We start with some auxiliary computations. By combining (2.19) and (6.71) we get, for $b = -a$,

$$\begin{aligned} B_{b,b} - A_{a,b} = &- \frac{a}{2\pi} tc \int_{-1}^{1} \int_{\mathbb{T}} \left(\frac{\tau^\perp}{(z_\lambda - z_b')^2} \right)_1 (\partial_\alpha z_b - \partial_\alpha z_b' - \beta \partial_\alpha^2 z_b) \, \mathrm{d}\beta \, \mathrm{d}\lambda \\ &- \frac{a}{2\pi} tc \partial_\alpha^2 z_b \sum_{j=0,1} \left(\frac{\tau^\perp}{(\partial_\alpha z_b)^{2-j}} \int_{-1}^{1} \left(\int_{\mathbb{T}} \beta^j \frac{\partial_\alpha z_b - \partial_\alpha z_b'}{(z_\lambda - z_b')^{j+1}} \, \mathrm{d}\beta + S_{\lambda,b}^j \right) \mathrm{d}\lambda \right)_1 \\ &+ \frac{a}{2} tc \partial_\alpha^2 z_b \left(\frac{\tau}{(\partial_\alpha z_b)^2} \int_{-1}^{1} (1 + \operatorname{sgn}(\lambda - b)) \, \mathrm{d}\lambda \right)_1, \end{aligned}$$

where $S_{\lambda,b}^j$ is given in (2.18). Notice that

$$\int_{-1}^{1} (1 + \operatorname{sgn}(\lambda - b)) \, \mathrm{d}\lambda = 2(1 + a).$$

Furthermore, it is easy to see that $S_{\lambda,b}^j - S_{\lambda,0}^j = O(t)$ by the regularity conditions (6.36)-(6.39). Then, by writing $\partial_\alpha z_b = \partial_\alpha z - itb\partial_\alpha (c\tau)$, it follows that

$$\begin{aligned} B_{b,b} - A_{a,b} = &- \frac{a}{2\pi} tc \int_{-1}^{1} \int_{\mathbb{T}} \left(\frac{\tau^\perp}{(z_\lambda - z_b')^2} \right)_1 (\partial_\alpha z - \partial_\alpha z' - \beta \partial_\alpha^2 z) \, \mathrm{d}\beta \, \mathrm{d}\lambda \\ &- \frac{a}{2\pi} tc \partial_\alpha^2 z \sum_{j=0,1} \left(\frac{\tau^\perp}{(\partial_\alpha z)^{2-j}} \int_{-1}^{1} \left(\int_{\mathbb{T}} \beta^j \frac{\partial_\alpha z - \partial_\alpha z'}{(z_\lambda - z_b')^{j+1}} \, \mathrm{d}\beta + S_{\lambda,0}^j \right) \mathrm{d}\lambda \right)_1 \\ &+ (1 + a) tc \partial_\alpha^2 z \left(\frac{\tau}{(\partial_\alpha z)^2} \right)_1 \\ &+ O(t^2 c). \end{aligned}$$

Therefore, analogously to (6.69)(6.70), the fundamental theorem of calculus jointly with the Fubini's theorem yield (recall (6.68))

$$
\begin{aligned}
E - B - t\kappa &= \tfrac{1}{2}(E - B_-) + \tfrac{1}{2}(E - B_+) - t\kappa \\
&= \frac{1}{\pi} t^2 c \int_{-1}^{1}\int_{-1}^{1}\int_{\mathbb{T}} \left(\frac{\tau c' \tau'}{(z_\lambda - z'_\mu)^3} \right)_1 (\partial_\alpha z - \partial_\alpha z' - \beta \partial_\alpha^2 z) \,\mathrm{d}\beta \,\mathrm{d}\lambda \,\mathrm{d}\mu && =: I_1 \\
&\quad - \frac{1}{2\pi} t^2 c \partial_\alpha^2 z \sum_{j=0,1} \left(\frac{(j+1)\tau}{(\partial_\alpha z)^{2-j}} \int_{-1}^{1}\int_{-1}^{1}\int_{\mathbb{T}} c'\tau'\beta^j \frac{\partial_\alpha z - \partial_\alpha z'}{(z_\lambda - z'_\mu)^{j+2}} \,\mathrm{d}\beta \,\mathrm{d}\lambda \,\mathrm{d}\mu \right)_1 && =: I_2 \\
&\quad - it\partial_\alpha(c\tau) \frac{1}{\pi} (C^1_{-,+} - C^1_{+,-})_1 - 2it\partial_\alpha(c\tau) \left(\frac{1}{\partial_\alpha z^\circ} \right)_2 && =: I_3 \\
&\quad + 2tc\partial_\alpha^2 z \left(\frac{\tau}{(\partial_\alpha z)^2} \right)_1 - 2tc\partial_\alpha^2 z^\circ \left(\frac{\tau}{(\partial_\alpha z^\circ)^2} \right)_1 && =: I_4 \\
&\quad + O(t^2 c).
\end{aligned}
$$

For I_1, by adding and subtracting $c\tau$ and $\frac{1}{2}\beta^2\partial_\alpha^3 z$, we split it into

$$
I_1 = -\frac{1}{2\pi}(tc)^2 \partial_\alpha^3 z \left(\tau^2 \int_{-1}^{1}\int_{-1}^{1} C^3_{\lambda,\mu} \,\mathrm{d}\lambda \,\mathrm{d}\mu \right)_1 + \text{commutators},
$$

where

$$
\begin{aligned}
\text{commutators} &= \frac{1}{\pi} t^2 c \int_{-1}^{1}\int_{-1}^{1}\int_{\mathbb{T}} \left(\frac{\tau((c\tau)' - c\tau)}{(z_\lambda - z'_\mu)^3} \right)_1 (\partial_\alpha z - \partial_\alpha z' - \beta\partial_\alpha^2 z) \,\mathrm{d}\beta \,\mathrm{d}\lambda \,\mathrm{d}\mu \\
&\quad + \frac{1}{\pi}(tc)^2 \int_{-1}^{1}\int_{-1}^{1}\int_{\mathbb{T}} \left(\frac{\tau^2}{(z_\lambda - z'_\mu)^3} \right)_1 (\partial_\alpha z - \partial_\alpha z' - \beta\partial_\alpha^2 z + \tfrac{1}{2}\beta^2\partial_\alpha^3 z) \,\mathrm{d}\beta \,\mathrm{d}\lambda \,\mathrm{d}\mu.
\end{aligned}
$$

The first term of I_1 is $O((tc)^2)$ by Lemma 6.4.1 and the commutators are $O(t^2 c)$ in terms of $\|z\|_{C_t C^{3,\delta}} \lesssim \|z\|_{C_t H^4}$. Similarly, by adding and subtracting $c\tau$ and $\beta\partial_\alpha^2 z$ for I_2, we gain commutators of order $O(t^2 c)$, while the remaining term reads as

$$
-\frac{1}{2\pi}(tc)^2 \partial_\alpha^2 z \sum_{j=0,1} \left(\frac{(j+1)\tau^2\partial_\alpha^2 z}{(\partial_\alpha z)^{2-j}} \int_{-1}^{1}\int_{-1}^{1} C^{j+2}_{\lambda,\mu} \,\mathrm{d}\lambda \,\mathrm{d}\mu \right)_1 = O((tc)^2),
$$

where we have applied Lemma 6.4.1. For I_3, (2.19) yields

$$
C^1_{\lambda,\mu} = \frac{1}{\partial_\alpha z_\mu} \left(\int_{\mathbb{T}} \frac{\partial_\alpha z_\mu - \partial_\alpha z'_\mu}{z_\lambda - z'_\mu} \,\mathrm{d}\beta + (1 + \operatorname{sgn}(\lambda - \mu))\pi i \right),
$$

and thus

$$
\frac{1}{\pi}(C^1_{-,+} - C^1_{+,-})_1 = O(t) - 2\left(\frac{1}{\partial_\alpha z_\mu} \right)_2 .
$$

Finally, a direct computation shows that

$$
\left(\frac{1}{\partial_\alpha z_\mu} \right)_2 - \left(\frac{1}{\partial_\alpha z^\circ} \right)_2 = O(t),
$$

and also for I_4

$$\partial_\alpha^2 z \left(\frac{\tau}{(\partial_\alpha z)^2}\right)_1 - \partial_\alpha^2 z^\circ \left(\frac{\tau}{(\partial_\alpha z^\circ)^2}\right)_1 = O(t),$$

in terms of $\|\partial_t z\|_{C_t C^{2,0}} \lesssim \|\partial_t z\|_{C_t H^3}$. $\square$

The next lemmas deal with D, h and $E^{1)}$.

Lemma 6.4.4. *For $0 \leq t \leq T$ it holds that*

$$D - D^{(0)} \partial_\alpha z \cdot \partial_\alpha z^\circ = O(t).$$

Proof. Recall that $D = -\frac{1}{2}(B_+ - B_-) + D^{(0)}$. Then, the statement follows from Lemma 6.4.2 since

$$B_+ - B_- = (B_+ - E) + (E - B_-) = O(t),$$

and using that $\partial_\alpha z \cdot \partial_\alpha z^\circ = 1 + O(t)$. $\square$

Lemma 6.4.5. *For $0 \leq t \leq T$ it holds that*

$$h, E^{1)} - E = O(t^2).$$

Proof. Recall the definition of $E^{(0)}$ and $E^{(1)}$ in (6.62). First of all we observe that, in analogy with the Hilbert transform, it follows that $E^{(0)} \in H^{k_\circ - 1}$ and also $E^{(1)} \in H^{k_\circ - 2}$ in terms of the chord-arc constant $\mathcal{C}(z^\circ)$ and $\|z^\circ\|_{H^{k_\circ}}$. Similarly, by differentiating E in time

$$\begin{aligned} \partial_t E = {} & \frac{1}{2\pi} \sum_{b=\pm} \int_{\mathbb{T}} \left(\frac{1}{z_b(t,\alpha) - z_b(t,\beta)}\right)_1 (\partial_\alpha \partial_t z_b(t,\alpha) - \partial_\alpha \partial_t z_b(t,\beta)) \, \mathrm{d}\beta \\ & - \frac{1}{2\pi} \sum_{b=\pm} \int_{\mathbb{T}} \left(\frac{\partial_t z_b(t,\alpha) - \partial_t z_b(t,\beta)}{(z_b(t,\alpha) - z_b(t,\beta))^2}\right)_1 (\partial_\alpha z_b(t,\alpha) - \partial_\alpha z_b(t,\beta)) \, \mathrm{d}\beta, \end{aligned} \tag{6.72}$$

Theorem 6.4.1 provides enough regularity (recall $k_\circ \geq 6$) and validity of chord-arc condition to obtain that $\partial_t E \in C_t H^{k_\circ - 4}$ as long as $0 \leq t \leq T$. In particular, since $E^{(0)} = E|_{t=0}$, the mean value theorem yields

$$\|E - E^{(0)}\|_{C_t H^{k_\circ - 4}} = O(t). \tag{6.73}$$

Hence, recalling the definition of h (6.64) together with Lemmas 6.4.3 and 6.4.4, it follows that $h = O(t)$ as well. Then, by writing

$$F - E^{(0)} = \psi_0(E - E^{(0)}) + t\psi_1 E^{(1)} - (t\kappa + i(tD^{(0)} \cdot \partial_\alpha(c\tau) + h\psi_1)\partial_\alpha z^\circ),$$

it follows from (6.73) and the regularity of the remaining terms that $\|F - E^{(0)}\|_{C_t H^{k_\circ - 4}} = O(t)$. As a result from (6.61), (6.62) and (6.72), we have $\partial_t E - E^{(1)} = O(t)$. Notice this implies $E^{(1)} = \partial_t E|_{t=0}$. Therefore, by applying the fundamental theorem of calculus

$$E - E^{1)} = \int_0^t (\partial_t E(s) - E^{(1)}) \, \mathrm{d}s,$$

we get $E - E^{1)} = O(t^2)$. Finally, this is enough to update the estimate for h to $O(t^2)$ as well. $\square$

Proof of Theorem 6.4.1. Proof of (6.48a). On $tc(\alpha) = 0$ we directly have that $\partial_t z = E = B$. *Proof of* (6.51). Consider now $tc(\alpha) > 0$. For (6.51a), the expression (6.65) and a direct use of Lemmas 6.4.2 and 6.4.5 yield that

$$\partial_t z - B_a = O(t).$$

For (6.51b), we use the expression (6.66). Then, Lemmas 6.4.3-6.4.5 imply that

$$\begin{aligned}
\left| \int_{\alpha_1}^{\alpha} (E - B - t\kappa) \cdot \partial_\alpha z^\perp \, \mathrm{d}\alpha' \right| &\lesssim t^2 \int_{\alpha_1}^{\alpha} (c + |\partial_\alpha c|) \, \mathrm{d}\alpha', \\
\left| \int_{\alpha_1}^{\alpha} \psi_1 (E^{1)} - E) \cdot \partial_\alpha z^\perp \, \mathrm{d}\alpha' \right| &\lesssim t^2 \int_{\alpha_1}^{\alpha} \psi_1 \, \mathrm{d}\alpha', \\
\left| \int_{\alpha_1}^{\alpha} t(D - D^{(0)} \partial_\alpha z \cdot \partial_\alpha z^\circ) \cdot \partial_\alpha (c\tau) \, \mathrm{d}\alpha' \right| &\lesssim t^2 \int_{\alpha_1}^{\alpha} (c + |\partial_\alpha c|) \, \mathrm{d}\alpha', \\
\left| \int_{\alpha_1}^{\alpha} h\psi_1 \partial_\alpha z \cdot \partial_\alpha z^\circ \, \mathrm{d}\alpha' \right| &\lesssim t^2 \int_{\alpha_1}^{\alpha} \psi_1 \, \mathrm{d}\alpha',
\end{aligned}$$

uniformly on $c(\alpha) > 0$. Firstly, recall that $\psi_1 \lesssim c$ by Lemma 6.3.1. secondly, if α is closer to α_1, Corollary 6.3.1 controls the integrals $\int_{\alpha_1}^{\alpha} c \, \mathrm{d}\alpha'$ and $\int_{\alpha_1}^{\alpha} |\partial_\alpha c| \, \mathrm{d}\alpha'$ in terms of $c(\alpha)$. Hence, the four terms above are $O(t^2 c(\alpha))$.

If α is closer to α_2, Corollary 6.3.1 yields control on $\int_{\alpha}^{\alpha_2} c \, \mathrm{d}\alpha'$ and $\int_{\alpha}^{\alpha_2} |\partial_\alpha c| \, \mathrm{d}\alpha'$ in terms of $c(\alpha)$. However, by (6.64), it holds that

$$\int_{\alpha_1}^{\alpha} (H - h\psi_1 \partial_\alpha z \cdot \partial_\alpha z^\circ) \, \mathrm{d}\alpha' = - \int_{\alpha}^{\alpha_2} (H - h\psi_1 \partial_\alpha z \cdot \partial_\alpha z^\circ) \, \mathrm{d}\alpha',$$

and thus we can integrate on (α, α_2). Therefore, for all $\alpha \in (\alpha_1, \alpha_2)$ the full expression is $O(t^2 c(\alpha))$. Finally, by dividing by $tc(\alpha)$ we have proven that (6.51b) holds. □

6.5 Existence of z

Theorem 6.5.1. *Let $z^\circ \in H^{k_\circ}(\mathbb{T}; \mathbb{R}^2)$ be a closed chord-arc curve with $k_\circ \geq 6$. Then, there exists $z \in C_t H^{k_\circ - 2}$ with $\partial_t z \in C_t H^{k_\circ - 3}$ solving* (6.61) *and satisfying* (6.37)-(6.39) *for some $0 < T \ll 1$ depending on the chord-arc constant $\mathcal{C}(z^\circ)$ and the norm $\|z^\circ\|_{H^{k_\circ}}$.*

Remark 6.5.1. The initial regularity required $k_\circ = 6$ is due to the fact that the energy estimates are easier in H^4, some estimates in the proof of Lemmas 6.4.3-6.4.5 and that the pseudo-interface loses two derivatives on the mixing region as in [70, 113].

We split the proof of this theorem into two parts. Firstly, we obtain a priori energy estimates for the equation (6.61). secondly, we explain briefly how (6.61) is regularized in order to use the a priori estimates to show the existence of the desired solution z.

6.5.1 A priori energy estimates

We will take our energy as

$$\mathcal{E}(z) := \|z\|_{H^{k_\circ - 2}}^2 + \mathcal{A}(z)^{-1} + \mathcal{C}(z) + \mathcal{S}(z)^{-1}, \tag{6.74}$$

where $\mathcal{A}(z)$, $\mathcal{C}(z)$ are the angle and the chord-arc constants given in (2.5), (2.6) respectively, and

$$\mathcal{S}(z) := \inf\{\sigma(\alpha) \,:\, \alpha \in \operatorname{supp}\psi_0\}$$

measures the RT-stability of z on $\operatorname{supp}\psi_0$ (recall $\sigma = (\rho_+ - \rho_-)\partial_\alpha z_1$ with $\rho_\pm = \pm 1$). Notice that $\mathcal{C}(z^\circ) < \infty$ by hypothesis and that $\mathcal{A}(z^\circ) = 1$, $\mathcal{S}(z^\circ) \geq 2\eta(1-2s) > 0$ by construction (recall Lemma 6.3.3). It turns out that $\frac{d}{dt}(\mathcal{A}(z)^{-1} + \mathcal{C}(z) + \mathcal{S}(z)^{-1})$ is a lower order term w.r.t. $\mathcal{E}(z)$. Analogous terms to $\mathcal{C}$ and $\mathcal{S}$ are rigorously analyzed in [37, 23]. The term $\mathcal{A}$ can be treated with a similar technique.

Next, we analyze the Sobolev norm of (6.74). Given $0 \leq k \leq k_\circ - 2$, we split

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\partial_\alpha^k z\|_{L^2}^2 &= \int_{\mathbb{T}} \partial_\alpha^k z \cdot \partial_\alpha^k F \,\mathrm{d}\alpha \\
&= \int_{\mathbb{T}} \partial_\alpha^k z \cdot \partial_\alpha^k (\psi_0 E)\,\mathrm{d}\alpha && =: I \\
&+ \int_{\mathbb{T}} \partial_\alpha^k z \cdot \partial_\alpha^k (\psi_1 E^{1)} - (t\kappa + itD^{(0)} \cdot \partial_\alpha(c\tau)\partial_\alpha z^\circ))\,\mathrm{d}\alpha && =: I_\circ \\
&- \int_{\mathbb{T}} \partial_\alpha^k z \cdot \partial_\alpha^k (i\psi_1 \partial_\alpha z^\circ) h \,\mathrm{d}\alpha && =: I_h
\end{aligned}$$

We claim that the terms I, $I_\circ$ and I_h are controlled from above in terms of $\|z^\circ\|_{H^{k_\circ}}$, $\|c\|_{H^{k_\circ-1}}$, $\|\psi_j\|_{H^{k_\circ-2}}$ for $j = 0, 1$, $\|z\|_{H^{k_\circ-2}}$, $\mathcal{A}(z)^{-1}$, $\mathcal{C}(z)$ and $\mathcal{S}(z)^{-1}$.

The term $I_\circ$ is controlled because ψ_1, $E^{1)}$, κ, $D^{(0)}$, c and τ only depends on z°. Indeed, it is clear that ψ_1 and c are smooth by definition (6.59)(6.60), while τ and $D^{(0)}$ lose one derivative and κ loses two (recall (6.12)(6.63)). In analogy with the Hilbert transform (recall (6.62)) it follows that $E^{(0)}$ loses one derivative and similarly $E^{(1)}$ loses two, namely

$$\|E^{1)}\|_{H^{k_\circ-2}} \lesssim \|z^\circ\|_{H^{k_\circ}} + \|z^\circ\|_{H^{k_\circ}}^q,$$

in terms of $\mathcal{C}(z^\circ)$ and $\|c\|_{H^{k_\circ-1}}$, for some $q \in \mathbb{N}$.

The term I_h is controlled because $h(t)$ does not depend on α. As we saw in Lemmas 6.4.2-6.4.5, it follows that $\|h\|_{L^\infty}$ is controlled in terms of $\|z\|_{H^3}$, $\|z^\circ\|_{H^4}$, $\mathcal{A}(c,z)^{-1}$ and $\mathcal{C}(c,z)$. These quantities (6.38) and (6.39) are controlled by $\mathcal{A}(z)^{-1}$ and $\mathcal{C}(z)$ for small times (cf. Lemma 2.1.1).

The term I is expected to be controlled because at the linear level

$$\psi_0 E \sim -\sum_{b=\pm} \psi_0 \sigma_b \Lambda z_b,$$

where $\Lambda := (-\Delta)^{1/2}$ and $\sigma_b = (\rho_+ - \rho_-)(\partial_\alpha z_b)_1$, which satisfies $\psi_0\sigma_b \geq 0$ for small times as our energy controls $\mathcal{S}(z)^{-1}$ (see the next subsection for a detailed explanation).

Analysis of I

As we mentioned in the introduction, the analysis of I is classical for curves in the fully stable regime. In our case, all the terms are treated classically but the most singular one which needs further analysis. Let us present here the estimate for the main term and discuss the rest in the appendix. We will assume w.l.o.g. that $\mathbb{T} = [-\pi, \pi]$ ($\ell_\circ = 2\pi$).

The most singular term in I, for each $b = \pm$, is (recall sec. 6.1 Notation)

$$J := \frac{1}{2\pi} \int_{\mathbb{T}} \psi_0 \partial_\alpha^k z \cdot \int_{\mathbb{T}} \left(\frac{1}{\delta_\beta z_b} \right)_1 \partial_\alpha^{k+1} \delta_\beta z_b \, \mathrm{d}\beta \, \mathrm{d}\alpha.$$

Since $z_b = z - ibtc\tau$, the term with $\partial_\alpha^{k+1}\delta_\beta(c\tau)$ is controlled by $\mathcal{C}(z)$ and $\|c\tau\|_{H^{k+1}}$. For $\partial_\alpha^{k+1}\delta_\beta z$, by adding and subtracting a suitable term, we split it into

$$J_\sigma + J_\Phi := -\frac{1}{2} \int_{\mathbb{T}} \left(\frac{\psi_0}{\partial_\alpha z_b} \right)_1 \partial_\alpha^k z \cdot \Lambda(\partial_\alpha^k z) \, \mathrm{d}\alpha + \int_{\mathbb{T}} \psi_0 \partial_\alpha^k z \cdot \int_{\mathbb{T}} \Phi_b \partial_\alpha^{k+1} \delta_\beta z \, \mathrm{d}\beta \, \mathrm{d}\alpha,$$

where $\Lambda = (-\Delta)^{1/2} : H^1 \to L^2$ is the operator

$$\Lambda f(\alpha) := \frac{1}{2\pi} \mathrm{pv}\int_{\mathbb{T}} \frac{\partial_\beta f(\beta)}{\tan(\frac{\alpha-\beta}{2})} \, \mathrm{d}\beta = \frac{1}{4\pi} \mathrm{pv}\int_{\mathbb{T}} \frac{f(\alpha) - f(\beta)}{\sin^2(\frac{\alpha-\beta}{2})} \, \mathrm{d}\beta,$$

and Φ_b is the bounded kernel

$$\Phi_b(t, \alpha, \beta) := \frac{1}{2\pi} \left(\frac{1}{\delta_\beta z_b} - \frac{1}{\partial_\alpha z_b (2\tan(\beta/2))} \right)_1. \tag{6.75}$$

For J_σ we proceed as follows. Recall that $\partial_\alpha z_1^\circ > 0$ uniformly on $\operatorname{supp}\psi_0$. Indeed, this is why we have opened the mixing zone slightly inside the stable regime. Our energy (6.74) allows to assume that $(\partial_\alpha z_b)_1 > 0$ on $\operatorname{supp}\psi_0$ for later times. Hence, using the Córdoba-Córdoba pointwise inequality $2f \cdot \Lambda f \geq \Lambda(|f|^2)$ (see [36]) and the fact that Λ is self-adjoint, we deduce that

$$\begin{aligned} 4J_\sigma &\leq -\int_{\mathbb{T}} \left(\frac{\psi_0}{\partial_\alpha z_b} \right)_1 \Lambda(|\partial_\alpha^k z|^2) \, \mathrm{d}\alpha \\ &= -\int_{\mathbb{T}} \Lambda \left(\frac{\psi_0}{\partial_\alpha z_b} \right)_1 |\partial_\alpha^k z|^2 \, \mathrm{d}\alpha \leq \left\| \Lambda \left(\frac{\psi_0}{\partial_\alpha z_b} \right)_1 \right\|_{L^\infty} \|\partial_\alpha^k z\|_{L^2}^2, \end{aligned}$$

with the first term controlled by $\mathcal{C}(z)$ and $\|z_b\|_{C^{2,\delta}}$. Notice that, since the evolution of $(\partial_\alpha z_b)_1$ is controlled in terms of our energy $\mathcal{E}$, the time of positiveness of $(\partial_\alpha z_b)_1$ on $\operatorname{supp}\psi_0$ depends just on the initial data z°.

The estimate of J_Φ is classical. We present it here for completeness. By writing $\partial_\alpha^{k+1}\delta_\beta z_b = \partial_\alpha \partial_\alpha^k z_b + \partial_\beta \partial_\alpha^k z_b'$, we split

$$\begin{aligned} J_\Phi = &-\frac{1}{2} \int_{\mathbb{T}} |\partial_\alpha^k z|^2 \partial_\alpha \left(\psi_0 \int_{\mathbb{T}} \Phi_b \, \mathrm{d}\beta \right) \mathrm{d}\alpha &&=: L_1 \\ &-\int_{\mathbb{T}} \psi_0 \partial_\alpha^k z \cdot \left(\int_{\mathbb{T}} \partial_\alpha^k z_b' \partial_\beta \Phi_b \, \mathrm{d}\beta \right) \mathrm{d}\alpha &&=: L_2 \end{aligned}$$

where we have integrated by parts w.r.t. α and β for L_1 and L_2 respectively. On the one hand, L_1 is controlled because we can write

$$\int_{\mathbb{T}} \Phi_b \, \mathrm{d}\beta = \frac{1}{2\pi} \left(\mathrm{pv}\int_{\mathbb{T}} \frac{\mathrm{d}\beta}{\delta_\beta z_b} \right)_1 = \frac{1}{2\pi} \left(\frac{1}{\partial_\alpha z_b} \left(\int_{\mathbb{T}} \frac{\delta_\beta \partial_\alpha z_b}{\delta_\beta z_b} \, \mathrm{d}\beta + \underbrace{\mathrm{pv}\int_{\mathbb{T}} \frac{\partial_\alpha z_b'}{\delta_b z_b} \, \mathrm{d}\beta}_{=\pi i} \right) \right)_1,$$

which is bounded in C^1 by $\mathcal{C}(z)$ and $\|z_b\|_{C^{2,\delta}}$. On the other hand, L_2 is controlled because $\partial_\beta \Phi_b$ is bounded in terms of $\mathcal{C}(z)$ and $\|z_b\|_{C^3}$.

The analysis of the remaining terms is standard (see e.g. [37]). For completeness, we have presented a compact version in Lemma 6.8.1.

6.5.2 Regularization

In order to be able to apply the Picard's theorem we regularize the equation (6.61) via

$$\begin{aligned} \partial_t z &= \phi_\varepsilon * F_\varepsilon(t, z^\circ, z), \\ z|_{t=0} &= z^\circ, \end{aligned} \tag{6.76}$$

in terms of the parameter $\varepsilon > 0$, where

$$F_\varepsilon := \psi_0 E_\varepsilon + \psi_1 E^{1)} - (t\kappa + i(tD^{(0)} \cdot \partial_\alpha(c\tau) + h\psi_1)\partial_\alpha z^\circ),$$

which agrees with F except that E has been replaced by

$$E_\varepsilon(t, \alpha) := \frac{1}{2\pi} \sum_{b=\pm} \int_{\mathbb{T}} \left(\frac{1}{\delta_\beta z_b} \right)_1 \partial_\alpha \delta_\beta (\phi_\varepsilon * z_b) \, \mathrm{d}\beta.$$

Let us fix the open set where the Picard's theorem is applied. Firstly, let O_k be the open subset of H^k formed by chord-arc curves

$$O_k := \{z \in H^k(\mathbb{T}; \mathbb{R}^2) \, : \, \mathcal{C}(z) < \infty\}.$$

Secondly, given $z^\circ \in O_{k_\circ}$ and $k \leq k_\circ$, we define the open neighborhood $O_k(z^\circ)$ of z° in H^k as the set of curves $z \in O_k$ satisfying, for some fixed parameters $0 < A, C, R, S < \infty$,

$$\mathcal{A}(z) > A, \qquad \mathcal{C}(z) < C, \qquad \|z\|_{H^k} < R, \tag{6.77}$$

and also

$$\mathcal{S}(z) > S. \tag{6.78}$$

From left to right, the conditions in (6.77) establish that the angle between $\partial_\alpha z$ and τ is uniformly non-perpendicular, which is necessary for our construction of the mixing zone (cf. Rem. 6.2.3), and that the chord-arc constant and the H^k-norm of z are uniformly bounded respectively. Since we want $z^\circ \in O_k(z^\circ)$, necessarily $A < \mathcal{A}(z^\circ) = 1$, $C > \mathcal{C}(z^\circ)$ and $R > \|z^\circ\|_{H^k}$. The condition (6.78) establishes that z remains uniformly stable on $\operatorname{supp} \psi_0$. By section 6.3 (cf. Lemma 6.3.3) we consider $S < 2\eta(1 - 2s)$.

Lemma 6.5.1. *Assume that there exists $z^\varepsilon \in C([0, T_\varepsilon]; O_{k_\circ - 2}(z^\circ))$ solving (6.76) for some $0 < T_\varepsilon \ll 1$. Then, there exists $q \in \mathbb{N}$ satisfying*

$$\frac{d}{dt}\mathcal{E}(z^\varepsilon) \lesssim \mathcal{E}(z^\varepsilon) + \mathcal{E}(z^\varepsilon)^q, \tag{6.79}$$

in terms of A, C, R, S and $\|z^\circ\|_{H^{k_\circ}}$, but independently of ε.

Proof. This is totally analogous to the a priori energy estimates of the previous section (see [37]). □

Proof of Theorem 6.5.1. Step 1. Approximation sequence z^ε. For all $\varepsilon > 0$, a standard Picard iteration yields a time-dependent curve $z^\varepsilon \in C([0,T_\varepsilon]; O_{k_\circ-2}(z^\circ))$ satisfying

$$z^\varepsilon(t) = z^\circ + \int_0^t \phi_\varepsilon * F_\varepsilon(s, z^\circ, z^\varepsilon(s))\,\mathrm{d}s.$$

This T_ε is taken in terms of the parameters defining $O_{k_\circ-2}(z^\circ)$ in such a way that the conditions (2.7)(2.8) hold. Thus, the operators $B_{a,b}$ (and so h) are well defined.

As usual, Gronwall's inequality applied to (6.79) implies that $\|z^\varepsilon\|_{C([0,T_\varepsilon];H^{k_\circ-2})}$ is uniformly bounded in ε. Furthermore, z^ε satisfies (6.76) with $\|\partial_t z\|_{C([0,T_\varepsilon];H^{k_\circ-3})}$ uniformly bounded in ε. We notice that our system is not autonomous but smooth in time. All these facts guarantee that the times of existence T_ε do not vanish as $\varepsilon \to 0$, namely $T_\varepsilon \geq T > 0$.

Step 2. Convergence to z. By the Rellich-Kondrachov and the Banach-Alaoglu theorems, we may assume (taking a subsequence if necessary) that there exists $z \in C([0,T]; O_{k_\circ-2}(z^\circ))$ such that $z^\varepsilon \to z$ in $C_t H^{k_\circ-3}$ and also $\partial_\alpha^{k_\circ-2} z^\varepsilon \rightharpoonup \partial_\alpha^{k_\circ-2} z$ as $\varepsilon \to 0$. Furthermore, $\partial_t z \in C([0,T]; H^{k_\circ-3})$. Finally, it follow that z solves (6.61) and satisfies (6.37)-(6.39). □

6.6 Proof of the main results and generalizations

In this section we glue the several proofs of the previous sections to gain clarity of how the Theorems 6.1.1 and 6.1.2 are proved. In addition, we recall how this construction is generalized for piecewise constant coarse-grained densities.

Proof of Theorems 6.1.1 and 6.1.2

First of all we construct the growth-rate c and the partition of the unity $\{\psi_0, \psi_1\}$ as in (6.60) and (6.59) respectively, in terms of z° and some small parameters η, s (e.g. $\eta = s = \frac{1}{4}$, $\delta = 1$). By Lemma 6.3.2, this c satisfies the inequality (6.50) and the regularity condition (6.36) (indeed $c \in C^\infty$).

Once these functions are fixed, the Theorem 6.5.1 implies the existence of a time-dependent pseudo-interface z satisfying the equation (6.61) and the regularity conditions (6.37)-(6.39) for some $T \ll 1$. By Theorem 6.4.1, this z satisfies the growth conditions (6.48a)(6.51).

Next, we construct the mixing zone Ω_{mix} and the non-mixing zones $\Omega_\pm$ by (6.10)(6.34) respectively. Then, we define the triplet $(\bar\rho, \bar v, \bar m)$ by (6.41)(6.42)(6.47). Hence, by Proposition 6.2.3 and Lemma 6.2.3, $(\bar\rho, \bar v, \bar m)$ is a subsolution to IPM for some $0 < T' \leq T$.

Finally, the h-principle in IPM (Theorem 6.2.1) yields infinitely many mixing solutions to IPM starting from (6.1)(6.6).

The proof of Theorem 6.1.2 is analogous to the one of Theorem 6.1.1. The main difference for the asymptotically flat case is that, since the domain of integration is $\mathbb{R}$ instead of $\mathbb{T}$, most of the integrals are taken with the Cauchy's principal value at infinity. In this case, (6.43) reads as

$$\begin{aligned} \mathrm{Ind}_{z_+(t)}(x) &= \tfrac{1}{2}\mathbb{1}_{\Omega_+(t)}(x), \\ \mathrm{Ind}_{z_-(t)}(x) &= \tfrac{1}{2}(1 - \mathbb{1}_{\Omega_-(t)}(x)), \end{aligned} \tag{6.80}$$

which changes the limits (6.45) but not the result. The proof of Theorem 6.1.2 in the x_1-periodic case is even closer to the one of Theorem 6.1.1. In this case (6.80) holds as well, but we do not have to deal with the infinity since the domain is $\mathbb{T}$.

Piecewise constant coarse-grained densities

Following [70, 113] we split the mixing zone into several levels $L = \{\lambda_j : 1 \leq |j| \leq N\}$ for $N \geq 1$ with

$$\lambda_j = \mathrm{sgn} j \tfrac{2|j|-1}{2N-1},$$

namely we consider

$$\Omega^j_{\mathrm{mix}}(t) := \{z_\lambda(t,\alpha) : c(\alpha) > 0,\, \lambda \in (-\lambda_j, \lambda_j)\},$$

with z_λ defined as in (6.11), which satisfies $\Omega^1_{\mathrm{mix}} \subset \cdots \subset \Omega^N_{\mathrm{mix}} =: \Omega_{\mathrm{mix}}$. In addition we define $\Omega_\pm$ as in (6.34) (or (6.35)).

Analogously to [70, 113], we define the piecewise constant (coarse-grained) density as ($|L| = 2N$)

$$\bar\rho(t,x) := \frac{2}{|L|}\sum_{b\in L} \mathrm{Ind}_{z_b(t)}(x) - 1, \tag{6.81}$$

for the closed case (6.6), while for asymptotically flat curves (6.7) the definition (6.81) needs to remove the last -1. Observe that $\bar\rho = \pm 1$ on $\Omega_\pm$ while $\bar\rho$ approaches the linear profile in [126, 22, 26] inside the mixing zone.

Analogously to (6.42), the Biot-Savart law yields

$$\begin{aligned}\bar v(t,x) &= -\left(\frac{1}{\pi i|L|}\sum_{b\in L}\int \frac{(\partial_\alpha z_b(t,\beta))_2}{x - z_b(t,\beta)}\,\mathrm{d}\beta\right)^* \\ &= -\frac{1}{\pi|L|}\sum_{b\in L}\int\left(\frac{1}{x-z_b(t,\beta)}\right)_1 \partial_\alpha z_b(t,\beta)\,\mathrm{d}\beta, \qquad x \neq z_b(t,\beta).\end{aligned} \tag{6.82}$$

Analogously to (6.47), we write the relaxed momentum as

$$\bar m := \bar\rho\bar v - (1-\bar\rho^2)(\gamma + \tfrac{1}{2}i),$$

in terms of some

$$\gamma := \sum_{j=1}^N \nabla^\perp g_j \mathbb{1}_{\Omega^j_{\mathrm{mix}}},$$

with $g_j(t,x)$ to be determined. Hence, analogously to (6.56), we define g_j in (α,λ)-coordinates as

$$G_j(t,\alpha,\lambda) := \int_{\alpha_1}^{\alpha}\left(\sum_{a=\pm\lambda_j}\frac{\lambda+a}{2}(\partial_t z - B_a)\cdot\partial_\alpha z_a^\perp - \frac{1}{N}(\lambda_j c\tau + \tfrac{1}{2})\cdot\partial_\alpha z_\lambda\right)\mathrm{d}\alpha',$$

where

$$B_a := \sum_{b\in L} B_{a,b}.$$

Finally, using that

$$\frac{1}{N}\sum_{j=1}^{N}\lambda_j = \frac{N}{2N-1},$$

the condition $|\gamma| < \frac{1}{2}$ yields the more general regime for c given in (6.8) as $N \to \infty$ (cf. (6.55)). The rest follows analogously to the case $N = 1$ (see [70, 113]).

6.7 The pressure

Lemma 6.7.1. *Let (ρ, v) be a mixing solution from Theorem 6.1.1 or 6.1.2. Then, there exists a pressure p satisfying Darcy's law*

$$\int_0^t \int_{\mathbb{R}^2} ((v + \rho i)\cdot\Phi - p\nabla\cdot\Phi)\,\mathrm{d}x\,\mathrm{d}s = 0, \tag{6.83}$$

for every test function $\Phi \in C_c^2(\mathbb{R}^3;\mathbb{R}^2)$. Observe that (6.83) *agrees with* (6.28c) *for $\Phi = \nabla^\perp\phi$. Moreover, $v = \nabla^\perp\psi$ with p and ψ the continuous functions given by*

$$\begin{aligned}(p + i\psi)(t,x) &= \frac{1}{2\pi}\sum_{b=\pm}\int \log|x - z_b(t,\beta)|\partial_\alpha z_b(t,\beta)^*\,\mathrm{d}\beta \\ &\quad + \frac{1}{2\pi i}\int_{\Omega_{\mathrm{mix}}(t)} \frac{1}{x-y}(\rho - \bar\rho)(t,y)\,\mathrm{d}y.\end{aligned}$$

The first term corresponds to the macroscopic contribution of $\bar\rho$ (cf. (6.85)*). The second one is the fluctuation coming from $\rho - \bar\rho$ and vanishes outside Ω_{mix} (cf.* (6.84)*).*
Furthermore, for any fixed $E \in C(\mathbb{R}_+;\mathbb{R}_+)$ with $E(r) > 0$ for $r > 0$, we can select these (infinitely many) mixing solutions satisfying

$$|((p + i\psi) - (\bar p + i\bar\psi))(t,x)| \le E(\mathrm{dist}((t,x),\Omega_+\cup\Omega_-)), \tag{6.84}$$

where

$$(\bar p + i\bar\psi)(t,x) := \frac{1}{2\pi}\sum_{b=\pm}\int \log|x - z_b(t,\beta)|\partial_\alpha z_b(t,\beta)^*\,\mathrm{d}\beta. \tag{6.85}$$

Proof. Notice that $\bar v = \nabla^\perp\bar\psi$ by (6.42). In particular,

$$\nabla(\bar p + i\bar\psi) = -i\bar\rho,$$

in the sense of distributions. Following chapter 3 we consider the convex integration sequence $(\rho_k, v_k) \to (\rho, v)$ in $C_tL^\infty_{w^*}$. In fact, $\rho_k \to \rho$ in $C_tL^q_{\mathrm{loc}}$ for all $1 < q < \infty$ (see sec. 5.3). Let us split $\rho_k = \bar\rho + \rho_k'$, $v_k = \bar v + v_k'$ and $p_k := \bar p + p_k'$ for some p_k' to be determined. By construction (Lemma 5.3.1) $(\rho_k', v_k') = (\Delta\varphi_k', -\nabla^\perp\partial_1\varphi_k')$ for some real-valued function φ_k' which is smooth and compactly supported on Ω_{mix}. Notice that $v_k' = \nabla^\perp\psi_k'$ for $\psi_k' := -\partial_1\varphi_k'$. Hence, p_k satisfies $\nabla(p_k + i\psi_k) = -i\rho_k$ if and only if p_k' satisfies

$$\nabla(p_k' + i\psi_k') = -i\rho_k'.$$

Therefore ($\Delta = \nabla\nabla^*$)

$$p'_k + i\psi'_k = -i\nabla^*\varphi'_k + f_k,$$

for some (time-dependent) entire function f_k. Since φ'_k is compactly supported on Ω_{mix}, necessarily $(f_k(t,x))_2 \to 0$ as $|x| \to \infty$. Therefore, the Liouville's theorem ($e^{if_k(t)}$ is entire and bounded) implies that f_k equals a (time-dependent) real constant. Hence, as we are choosing p'_k, we may assume that $f_k = 0$. Finally, the Cauchy-Pompeiu's formula yields

$$\nabla^*\varphi'_k(t,x) = \frac{1}{2\pi}\int_{\Omega_{\mathrm{mix}}(t)} \frac{1}{x-y}\rho'_k(t,y)\,\mathrm{d}y.$$

This concludes the proof by taking the limit $k \to \infty$. The inequality (6.84) can be guaranteed by imposing it in the space of subsolutions $X_0(\bar{u},\mathcal{F})$ and then following the proof of the quantitative h-principle. □

6.8 Lower order terms

Lemma 6.8.1. *The remaining terms of I from section 6.5.1 are lower order terms.*

Proof. By combining the general Leibniz rule applied to $(\psi_0, K_b, \partial_\alpha\delta_\beta z_b)$ where

$$K_b(t,\alpha,\beta) := \left(\frac{1}{\delta_\beta z_b(t,\alpha)}\right)_1,$$

with the Faà di Bruno's formula applied to the kernel K_b, we split ($j = (j_0, j_1, j_2)$)

$$I = \frac{1}{2\pi}\sum_{b=\pm}\sum_{|j|=k}\sum_{n\in\pi_{j_1}}\binom{k}{j}(-1)^{|n|}C_n I_b(j,n),$$

where

$$I_b(j,n) := \int_{\mathbb{T}}(\partial_\alpha^{j_0}\psi_0)\partial_\alpha^k z\cdot\int_{\mathbb{T}}\left(\frac{\prod_{i=1}^{j_1}(\partial_\alpha^i\delta_\beta z_b)^{n_i}}{(\delta_\beta z_b)^{|n|+1}}\right)_1(\partial_\alpha^{j_2+1}\delta_\beta z_b)\,\mathrm{d}\beta\,\mathrm{d}\alpha,$$

with $\pi_{j_1} := \{n \in \mathbb{N}_0^{j_1} \,:\, n_1 + 2n_2 + \cdots + j_1 n_{j_1} = j_1\}$ and

$$C_n := \frac{|n|!j_1!}{n_1!1!^{n_1}\cdots n_{j_1}!j_1!^{n_{j_1}}} > 0.$$

The most singular term $j = (0,0,k)$ ($\Rightarrow n = 0$) has been analyzed in section 6.5.1.

Second singular term. Let us consider $j = (0,k,0)$. If $n_k = 1$ then

$$I_b(j,n) = \int_{\mathbb{T}}\psi_0\partial_\alpha^k z\cdot\int_{\mathbb{T}}\left(\frac{\partial_\alpha^k\delta_\beta z_b}{(\delta_\beta z_b)^2}\right)_1\partial_\alpha\delta_\beta z_b\,\mathrm{d}\beta\,\mathrm{d}\alpha.$$

By splitting $\partial_\alpha\delta_\beta z_b$ into its real and imaginary part and comparing

$$\frac{(\partial_\alpha\delta_\beta z_b)_l}{(\delta_\beta z_b)^2} \sim \frac{(\partial_\alpha^2 z_b)_l}{(\partial_\alpha z_b)^2(2\tan(\beta/2))}, \qquad l = 1,2,$$

we obtain a Hilbert transform acting on $\partial_\alpha^k z_b$ while the commutator is a bounded kernel as in (6.75). If $n_k = 0$, notice that for any $k \geq 3$ we have $n_{k-1} \leq \frac{k}{k-1} < 2$, that is $n_{k-1} = 0$ or 1 (the case $k < 3 < k_\circ - 2$ is easier). If $n_{k-1} = 1$ ($\Rightarrow n_1 = 1$) then simply

$$|I_b(j,n)| \lesssim \mathcal{C}(z)^3 \|\psi_0\|_{L^\infty} \|\partial_\alpha^k z\|_{L^2} |\partial_\alpha z_b|_{C^1}^2 |\partial_\alpha^{k-1} z_b|_{C^\delta},$$

and for $n_{k-1} = 0$

$$|I_b(j,n)| \lesssim \mathcal{C}(z)^{|n|+1} \|\psi_0\|_{L^\infty} \|\partial_\alpha^k z\|_{L^2} \|z_b\|_{C^{k-2,1}}^{|n|+1}. \tag{6.86}$$

Third singular term. Let us consider $j = (0, 1, k-1)$ ($\Rightarrow n = 1$):

$$I_b(j,n) = \int_{\mathbb{T}} \psi_0 \partial_\alpha^k z \cdot \int_{\mathbb{T}} \left(\frac{\partial_\alpha \delta_\beta z_b}{(\delta_\beta z_b)^2} \right)_1 (\partial_\alpha^k \delta_\beta z_b) \, \mathrm{d}\beta \, \mathrm{d}\alpha.$$

This is analogous to the case $(0, k, 0)$. The case $j = (1, 0, k-1)$ is analogous too. Let us consider now $j = (0, k-1, 1)$. If $n_{k-1} = 1$ then

$$I_b(j,n) = \int_{\mathbb{T}} \psi_0 \partial_\alpha^k z \cdot \int_{\mathbb{T}} \left(\frac{\partial_\alpha^{k-1} \delta_\beta z_b}{(\delta_\beta z_b)^2} \right)_1 (\partial_\alpha^2 \delta_\beta z_b) \, \mathrm{d}\beta \, \mathrm{d}\alpha,$$

and so

$$|I_b(j,n)| \lesssim \mathcal{C}(z)^2 \|\psi_0\|_{L^\infty} \|\partial_\alpha^k z\|_{L^2} |\partial_\alpha^{k-1} z_b|_{C^\delta} |\partial_\alpha^2 z_b|_{C^1}.$$

If $n_{k-1} = 0$ then (6.86) holds. The case $j = (1, k-1, 0)$ is analogous.

Harmless terms. For $0 \leq j_1, j_2 \leq k-2$, simply

$$|I_b(j,n)| \lesssim \mathcal{C}(z)^{|n|+1} \|\partial_\alpha^{j_0} \psi_0\|_{L^\infty} \|\partial_\alpha^k z\|_{L^2} \|z_b\|_{H^k}^{|n|+1}.$$

This concludes the proof. $\square$

Bibliography

[1] Ibrokhimbek Akramov and Emil Wiedemann. "NonUnique Admissible Weak Solutions of the Compressible Euler Equations with Compact Support in Space". In: *SIAM J. Math. Anal.* 53.1 (2021), pp. 795–812. ISSN: 0036-1410. DOI: 10.1137/20M1367015. URL: https://doi.org/10.1137/20M1367015.

[2] Thomas Alazard and Omar Lazar. "Paralinearization of the Muskat equation and application to the Cauchy problem". In: *Arch. Ration. Mech. Anal.* 237.2 (2020), pp. 545–583. ISSN: 0003-9527. DOI: 10.1007/s00205-020-01514-6. URL: https://doi.org/10.1007/s00205-020-01514-6.

[3] Thomas Alazard and Quoc-Hung Nguyen. "Endpoint Sobolev theory for the Muskat equation". In: *arXiv:2010.06915* (2020). URL: https://arxiv.org/abs/2010.06915.

[4] Thomas Alazard and Quoc-Hung Nguyen. "On the Cauchy problem for the Muskat equation. II: Critical initial data". In: *arXiv:2009.08442* (2020). URL: https://arxiv.org/abs/2009.08442.

[5] Thomas Alazard and Quoc-Hung Nguyen. "On the Cauchy problem for the Muskat equation with non-Lipschitz initial data". In: *arXiv:2009.04343* (2020). URL: https://arxiv.org/abs/2009.04343.

[6] David M. Ambrose. "The zero surface tension limit of two-dimensional interfacial Darcy flow". In: *J. Math. Fluid Mech.* 16.1 (2014), pp. 105–143. ISSN: 1422-6928. DOI: 10.1007/s00021-013-0146-1. URL: https://doi.org/10.1007/s00021-013-0146-1.

[7] David M. Ambrose. "Well-posedness of two-phase Hele-Shaw flow without surface tension". In: *European J. Appl. Math.* 15.5 (2004), pp. 597–607. ISSN: 0956-7925. DOI: 10.1017/S0956792504005662. URL: https://doi.org/10.1017/S0956792504005662.

[8] Víctor Arnaiz, Ángel Castro, and Daniel Faraco. "Semiclassical estimates for pseudodifferential operators and the Muskat problem in the unstable regime". In: *Comm. Partial Differential Equations* 46.1 (2021), pp. 135–164. ISSN: 0360-5302. DOI: 10.1080/03605302.2020.1831019. URL: https://doi.org/10.1080/03605302.2020.1831019.

[9] Rajendra Beekie, Tristan Buckmaster, and Vlad Vicol. "Weak solutions of ideal MHD which do not conserve magnetic helicity". In: *Ann. PDE* 6.1 (2020), Paper No. 1, 40. ISSN: 2524-5317. DOI: 10.1007/s40818-020-0076-1. URL: https://doi.org/10.1007/s40818-020-0076-1.

[10] Garrett Birkhoff. "Helmholtz and Taylor instability". In: *Proc. Sympos. Appl. Math., Vol. XIII.* American Mathematical Society, Providence, R.I., 1962, pp. 55–76.

[11] Yann Brenier, Camillo De Lellis, and László Székelyhidi Jr. "Weak-strong uniqueness for measure-valued solutions". In: *Comm. Math. Phys.* 305.2 (2011), pp. 351–361. ISSN: 0010-3616. DOI: 10.1007/s00220-011-1267-0. URL: https://doi.org/10.1007/s00220-011-1267-0.

[12] Tristan Buckmaster, Maria Colombo, and Vlad Vicol. "Wild solutions of the Navier-Stokes equations whose singular sets in time have Hausdorff dimension strictly less than 1". In: *J. Eur. Math. Soc. (JEMS)* (To appear).

[13] Tristan Buckmaster, Camillo de Lellis, László Székelyhidi Jr., and Vlad Vicol. "Onsager's conjecture for admissible weak solutions". In: *Comm. Pure Appl. Math.* 72.2 (2019), pp. 229–274. ISSN: 0010-3640. DOI: 10.1002/cpa.21781. URL: https://doi.org/10.1002/cpa.21781.

[14] Tristan Buckmaster, Steve Shkoller, and Vlad Vicol. "Nonuniqueness of weak solutions to the SQG equation". In: *Comm. Pure Appl. Math.* 72.9 (2019), pp. 1809–1874. ISSN: 0010-3640. DOI: 10.1002/cpa.21851. URL: https://doi.org/10.1002/cpa.21851.

[15] Tristan Buckmaster and Vlad Vicol. "Convex integration and phenomenologies in turbulence". In: *EMS Surv. Math. Sci.* 6.1-2 (2019), pp. 173–263. ISSN: 2308-2151. DOI: 10.4171/emss/34. URL: https://doi.org/10.4171/emss/34.

[16] Tristan Buckmaster and Vlad Vicol. "Nonuniqueness of weak solutions to the Navier-Stokes equation". In: *Ann. of Math. (2)* 189.1 (2019), pp. 101–144. ISSN: 0003-486X. DOI: 10.4007/annals.2019.189.1.3. URL: https://doi.org/10.4007/annals.2019.189.1.3.

[17] Jan Burczak, Stefano Modena, and László Székelyhidi Jr. "Non Uniqueness of power-law flows". In: *arXiv:2007.08011* (2020). URL: https://arxiv.org/abs/2007.08011.

[18] Russel E. Caflisch, Maria Carmela Lombardo, and Marco M. L. Sammartino. "Vortex layers of small thickness". In: *Comm. Pure Appl. Math.* 73.10 (2020), pp. 2104–2179. ISSN: 0010-3640. DOI: 10.1002/cpa.21897. URL: https://doi.org/10.1002/cpa.21897.

[19] Russel E. Caflisch and John S. Lowengrub. "Convergence of the vortex method for vortex sheets". In: *SIAM J. Numer. Anal.* 26.5 (1989), pp. 1060–1080. ISSN: 0036-1429. DOI: 10.1137/0726059. URL: https://doi.org/10.1137/0726059.

[20] Russel E. Caflisch and Oscar F. Orellana. "Singular solutions and ill-posedness for the evolution of vortex sheets". In: *SIAM J. Math. Anal.* 20.2 (1989), pp. 293–307. ISSN: 0036-1410. DOI: 10.1137/0520020. URL: https://doi.org/10.1137/0520020.

[21] Stephen Cameron. "Global well-posedness for the two-dimensional Muskat problem with slope less than 1". In: *Anal. PDE* 12.4 (2019), pp. 997–1022. ISSN: 2157-5045. DOI: 10.2140/apde.2019.12.997. URL: https://doi.org/10.2140/apde.2019.12.997.

[22] Ángel Castro, Diego Córdoba, and Daniel Faraco. "Mixing solutions for the Muskat problem". In: *Invent. Math.* (To appear).

[23] Ángel Castro, Diego Córdoba, Charles Fefferman, and Francisco Gancedo. "Breakdown of smoothness for the Muskat problem". In: *Arch. Ration. Mech. Anal.* 208.3 (2013), pp. 805–909. ISSN: 0003-9527. DOI: 10.1007/s00205-013-0616-x. URL: https://doi.org/10.1007/s00205-013-0616-x.

[24] Ángel Castro, Diego Córdoba, Charles Fefferman, Francisco Gancedo, and María López-Fernández. "Rayleigh-Taylor breakdown for the Muskat problem with applications to water waves". In: *Ann. of Math. (2)* 175.2 (2012), pp. 909–948. ISSN: 0003-486X. DOI: `10.4007/annals.2012.175.2.9`. URL: `https://doi.org/10.4007/annals.2012.175.2.9`.

[25] Ángel Castro, Diego Córdoba, and Francisco Gancedo. "A naive parametrization for the vortex-sheet problem". In: *Mathematical aspects of fluid mechanics*. Vol. 402. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2012, pp. 88–115.

[26] Ángel Castro, Daniel Faraco, and Francisco Mengual. "Degraded mixing solutions for the Muskat problem". In: *Calc. Var. Partial Differential Equations* 58.2 (2019), Paper No. 58, 29. ISSN: 0944-2669. DOI: `10.1007/s00526-019-1489-0`. URL: `https://doi.org/10.1007/s00526-019-1489-0`.

[27] Ángel Castro, Daniel Faraco, and Francisco Mengual. "Localized mixing zone for Muskat bubbles and turned interfaces". In: *arXiv:2102.07451* (2021). URL: `https://arxiv.org/abs/2102.07451`.

[28] C. H. Arthur Cheng, Rafael Granero-Belinchón, and Steve Shkoller. "Well-posedness of the Muskat problem with H^2 initial data". In: *Adv. Math.* 286 (2016), pp. 32–104. ISSN: 0001-8708. DOI: `10.1016/j.aim.2015.08.026`. URL: `https://doi.org/10.1016/j.aim.2015.08.026`.

[29] Elisabetta Chiodaroli, Camillo De Lellis, and Ondřej Kreml. "Global ill-posedness of the isentropic system of gas dynamics". In: *Comm. Pure Appl. Math.* 68.7 (2015), pp. 1157–1190. ISSN: 0010-3640. DOI: `10.1002/cpa.21537`. URL: `https://doi.org/10.1002/cpa.21537`.

[30] Elisabetta Chiodaroli and Ondřej Kreml. "On the energy dissipation rate of solutions to the compressible isentropic Euler system". In: *Arch. Ration. Mech. Anal.* 214.3 (2014), pp. 1019–1049. ISSN: 0003-9527. DOI: `10.1007/s00205-014-0771-8`. URL: `https://doi.org/10.1007/s00205-014-0771-8`.

[31] Maria Colombo, Camillo De Lellis, and Luigi De Rosa. "Ill-posedness of Leray solutions for the hypodissipative Navier-Stokes equations". In: *Comm. Math. Phys.* 362.2 (2018), pp. 659–688. ISSN: 0010-3616. DOI: `10.1007/s00220-018-3177-x`. URL: `https://doi.org/10.1007/s00220-018-3177-x`.

[32] Peter Constantin, Diego Córdoba, Francisco Gancedo, Luis Rodríguez-Piazza, and Robert M. Strain. "On the Muskat problem: global in time results in 2D and 3D". In: *Amer. J. Math.* 138.6 (2016), pp. 1455–1494. ISSN: 0002-9327. DOI: `10.1353/ajm.2016.0044`. URL: `https://doi.org/10.1353/ajm.2016.0044`.

[33] Peter Constantin, Diego Córdoba, Francisco Gancedo, and Robert M. Strain. "On the global existence for the Muskat problem". In: *J. Eur. Math. Soc. (JEMS)* 15.1 (2013), pp. 201–227. ISSN: 1435-9855. DOI: `10.4171/JEMS/360`. URL: `https://doi.org/10.4171/JEMS/360`.

[34] Peter Constantin, Weinan E, and Edriss S. Titi. "Onsager's conjecture on the energy conservation for solutions of Euler's equation". In: *Comm. Math. Phys.* 165.1 (1994), pp. 207–209. ISSN: 0010-3616. URL: `http://projecteuclid.org/euclid.cmp/1104271041`.

[35] Peter Constantin, Francisco Gancedo, Roman Shvydkoy, and Vlad Vicol. "Global regularity for 2D Muskat equations with finite slope". In: *Ann. Inst. H. Poincaré Anal. Non Linéaire* 34.4 (2017), pp. 1041–1074. ISSN: 0294-1449. DOI: 10.1016/j.anihpc.2016.09.001. URL: https://doi.org/10.1016/j.anihpc.2016.09.001.

[36] Antonio Córdoba and Diego Córdoba. "A pointwise estimate for fractionary derivatives with applications to partial differential equations". In: *Proc. Natl. Acad. Sci. USA* 100.26 (2003), pp. 15316–15317. ISSN: 0027-8424. DOI: 10.1073/pnas.2036515100. URL: https://doi.org/10.1073/pnas.2036515100.

[37] Antonio Córdoba, Diego Córdoba, and Francisco Gancedo. "Interface evolution: the Hele-Shaw and Muskat problems". In: *Ann. of Math. (2)* 173.1 (2011), pp. 477–542. ISSN: 0003-486X. DOI: 10.4007/annals.2011.173.1.10. URL: https://doi.org/10.4007/annals.2011.173.1.10.

[38] Antonio Córdoba, Diego Córdoba, and Francisco Gancedo. "Uniqueness for SQG patch solutions". In: *Trans. Amer. Math. Soc. Ser. B* 5 (2018), pp. 1–31. DOI: 10.1090/btran/20. URL: https://doi.org/10.1090/btran/20.

[39] Diego Córdoba, Daniel Faraco, and Francisco Gancedo. "Lack of uniqueness for weak solutions of the incompressible porous media equation". In: *Arch. Ration. Mech. Anal.* 200.3 (2011), pp. 725–746. ISSN: 0003-9527. DOI: 10.1007/s00205-010-0365-z. URL: https://doi.org/10.1007/s00205-010-0365-z.

[40] Diego Córdoba and Francisco Gancedo. "Contour dynamics of incompressible 3-D fluids in a porous medium with different densities". In: *Comm. Math. Phys.* 273.2 (2007), pp. 445–471. ISSN: 0010-3616. DOI: 10.1007/s00220-007-0246-y. URL: https://doi.org/10.1007/s00220-007-0246-y.

[41] Diego Córdoba and Omar Lazar. "Global well-posedness for the 2D stable Muskat problem in $H^{3/2}$". In: *Ann. Sci. Éc. Norm. Supér.* (To appear).

[42] Diego Córdoba Gazolaz, Rafael Granero-Belinchón, and Rafael Orive-Illera. "The confined Muskat problem: differences with the deep water regime". In: *Commun. Math. Sci.* 12.3 (2014), pp. 423–455. ISSN: 1539-6746. DOI: 10.4310/CMS.2014.v12.n3.a2. URL: https://doi.org/10.4310/CMS.2014.v12.n3.a2.

[43] Gianluca Crippa, Nikolay Gusev, Stefano Spirito, and Emil Wiedemann. "Non-uniqueness and prescribed energy for the continuity equation". In: *Commun. Math. Sci.* 13.7 (2015), pp. 1937–1947. ISSN: 1539-6746. DOI: 10.4310/CMS.2015.v13.n7.a12. URL: https://doi.org/10.4310/CMS.2015.v13.n7.a12.

[44] Benoit Cushman-Roisin and Jean-Marie Beckers. *Introduction to geophysical fluid dynamics*. Second. Vol. 101. International Geophysics Series. Physical and numerical aspects, With a foreword by John Marshall. Elsevier/Academic Press, Amsterdam, 2011, pp. xviii+828. ISBN: 978-0-12-088759-0.

[45] Bernard Dacorogna and Paolo Marcellini. "General existence theorems for Hamilton-Jacobi equations in the scalar and vectorial cases". In: *Acta Math.* 178.1 (1997), pp. 1–37. ISSN: 0001-5962. DOI: 10.1007/BF02392708. URL: https://doi.org/10.1007/BF02392708.

[46] Constantine M. Dafermos. "The entropy rate admissibility criterion for solutions of hyperbolic conservation laws". In: *J. Differential Equations* 14 (1973), pp. 202–212. ISSN: 0022-0396. DOI: `10.1016/0022-0396(73)90043-0`. URL: `https://doi.org/10.1016/0022-0396(73)90043-0`.

[47] Sara Daneri, Eris Runa, and László Székelyhidi Jr. "Non-uniqueness for the Euler equations up to Onsager's critical exponent". In: *arXiv:2004.00391* (2020). URL: `https://arxiv.org/abs/2004.00391`.

[48] Sara Daneri and László Székelyhidi Jr. "Non-uniqueness and h-principle for Hölder-continuous weak solutions of the Euler equations". In: *Arch. Ration. Mech. Anal.* 224.2 (2017), pp. 471–514. ISSN: 0003-9527. DOI: `10.1007/s00205-017-1081-8`. URL: `https://doi.org/10.1007/s00205-017-1081-8`.

[49] Henry Philibert Gaspard Darcy. *Les fontaines publiques de la ville de Dijon. Exposition et application des principes à suivre et des formules à employer dans les questions de distribution d'eau.* Paris: Dalmont, V., Ed., 1856.

[50] Guy David. "Opérateurs intégraux singuliers sur certaines courbes du plan complexe". In: *Ann. Sci. École Norm. Sup. (4)* 17.1 (1984), pp. 157–189. ISSN: 0012-9593. URL: `http://www.numdam.org/item?id=ASENS_1984_4_17_1_157_0`.

[51] Camillo De Lellis and László Székelyhidi Jr. "Dissipative continuous Euler flows". In: *Invent. Math.* 193.2 (2013), pp. 377–407. ISSN: 0020-9910. DOI: `10.1007/s00222-012-0429-9`. URL: `https://doi.org/10.1007/s00222-012-0429-9`.

[52] Camillo De Lellis and László Székelyhidi Jr. "High dimensionality and h-principle in PDE". In: *Bull. Amer. Math. Soc. (N.S.)* 54.2 (2017), pp. 247–282. ISSN: 0273-0979. DOI: `10.1090/bull/1549`. URL: `https://doi.org/10.1090/bull/1549`.

[53] Camillo De Lellis and László Székelyhidi Jr. "On admissibility criteria for weak solutions of the Euler equations". In: *Arch. Ration. Mech. Anal.* 195.1 (2010), pp. 225–260. ISSN: 0003-9527. DOI: `10.1007/s00205-008-0201-x`. URL: `https://doi.org/10.1007/s00205-008-0201-x`.

[54] Camillo De Lellis and László Székelyhidi Jr. "The Euler equations as a differential inclusion". In: *Ann. of Math. (2)* 170.3 (2009), pp. 1417–1436. ISSN: 0003-486X. DOI: `10.4007/annals.2009.170.1417`. URL: `https://doi.org/10.4007/annals.2009.170.1417`.

[55] Camillo De Lellis and László Székelyhidi Jr. "The h-principle and the equations of fluid dynamics". In: *Bull. Amer. Math. Soc. (N.S.)* 49.3 (2012), pp. 347–375. ISSN: 0273-0979. DOI: `10.1090/S0273-0979-2012-01376-9`. URL: `https://doi.org/10.1090/S0273-0979-2012-01376-9`.

[56] Jean-Marc Delort. "Existence de nappes de tourbillon en dimension deux". In: *J. Amer. Math. Soc.* 4.3 (1991), pp. 553–586. ISSN: 0894-0347. DOI: `10.2307/2939269`. URL: `https://doi.org/10.2307/2939269`.

[57] Fan Deng, Zhen Lei, and Fanghua Lin. "On the two-dimensional Muskat problem with monotone large initial data". In: *Comm. Pure Appl. Math.* 70.6 (2017), pp. 1115–1145. ISSN: 0010-3640. DOI: `10.1002/cpa.21669`. URL: `https://doi.org/10.1002/cpa.21669`.

[58] Eleonora Di Nezza, Giampiero Palatucci, and Enrico Valdinoci. “Hitchhiker’s guide to the fractional Sobolev spaces”. In: *Bull. Sci. Math.* 136.5 (2012), pp. 521–573. ISSN: 0007-4497. DOI: 10.1016/j.bulsci.2011.12.004. URL: https://doi.org/10.1016/j.bulsci.2011.12.004.

[59] Jean Duchon and Raoul Robert. “Global vortex sheet solutions of Euler equations in the plane”. In: *J. Differential Equations* 73.2 (1988), pp. 215–224. ISSN: 0022-0396. DOI: 10.1016/0022-0396(88)90105-2. URL: https://doi.org/10.1016/0022-0396(88)90105-2.

[60] Jean Duchon and Raoul Robert. “Inertial energy dissipation for weak solutions of incompressible Euler and Navier-Stokes equations”. In: *Nonlinearity* 13.1 (2000), pp. 249–255. ISSN: 0951-7715. DOI: 10.1088/0951-7715/13/1/312. URL: https://doi.org/10.1088/0951-7715/13/1/312.

[61] David G. Ebin. “Ill-posedness of the Rayleigh-Taylor and Helmholtz problems for incompressible fluids”. In: *Comm. Partial Differential Equations* 13.10 (1988), pp. 1265–1295. ISSN: 0360-5302. DOI: 10.1080/03605308808820576. URL: https://doi.org/10.1080/03605308808820576.

[62] Leonhard Euler. “Principes généraux du mouvement des fluides”. In: *Mémoires de l’académie des sciences de Berlin* 11 (1757), pp. 274–315.

[63] Lawrence C. Evans and Stefan Müller. “Hardy spaces and the two-dimensional Euler equations with nonnegative vorticity”. In: *J. Amer. Math. Soc.* 7.1 (1994), pp. 199–219. ISSN: 0894-0347. DOI: 10.2307/2152727. URL: https://doi.org/10.2307/2152727.

[64] Gregory L. Eyink. “Energy dissipation without viscosity in ideal hydrodynamics. I. Fourier analysis and local energy transfer”. In: *Phys. D* 78.3-4 (1994), pp. 222–240. ISSN: 0167-2789. DOI: 10.1016/0167-2789(94)90117-1. URL: https://doi.org/10.1016/0167-2789(94)90117-1.

[65] Daniel Faraco and Sauli Lindberg. “Proof of Taylor’s conjecture on magnetic helicity conservation”. In: *Comm. Math. Phys.* 373.2 (2020), pp. 707–738. ISSN: 0010-3616. DOI: 10.1007/s00220-019-03422-7. URL: https://doi.org/10.1007/s00220-019-03422-7.

[66] Daniel Faraco, Sauli Lindberg, and László Székelyhidi. “Bounded Solutions of Ideal MHD with Compact Support in Space-Time”. In: *Arch. Ration. Mech. Anal.* 239.1 (2021), pp. 51–93. ISSN: 0003-9527. DOI: 10.1007/s00205-020-01570-y. URL: https://doi.org/10.1007/s00205-020-01570-y.

[67] Eduard Feireisl. “Maximal dissipation and well-posedness for the compressible Euler system”. In: *J. Math. Fluid Mech.* 16.3 (2014), pp. 447–461. ISSN: 1422-6928. DOI: 10.1007/s00021-014-0163-8. URL: https://doi.org/10.1007/s00021-014-0163-8.

[68] Eduard Feireisl, Christian Klingenberg, and Simon Markfelder. “On the density of “wild” initial data for the compressible Euler system”. In: *Calc. Var. Partial Differential Equations* 59.5 (2020), Paper No. 152, 17. ISSN: 0944-2669. DOI: 10.1007/s00526-020-01806-5. URL: https://doi.org/10.1007/s00526-020-01806-5.

[69] Gerald B. Folland. *Introduction to partial differential equations*. Second. Princeton University Press, Princeton, NJ, 1995, pp. xii+324. ISBN: 0-691-04361-2.

[70] Clemens Förster and László Székelyhidi Jr. “Piecewise constant subsolutions for the Muskat problem”. In: *Comm. Math. Phys.* 363.3 (2018), pp. 1051–1080. ISSN: 0010-3616. DOI: 10.1007/s00220-018-3245-2. URL: https://doi.org/10.1007/s00220-018-3245-2.

[71] Uriel Frisch. *Turbulence*. The legacy of A. N. Kolmogorov. Cambridge University Press, Cambridge, 1995, pp. xiv+296. ISBN: 0-521-45103-5.

[72] Francisco Gancedo. “A survey for the Muskat problem and a new estimate”. In: *SeMA J.* 74.1 (2017), pp. 21–35. ISSN: 2254-3902. DOI: 10.1007/s40324-016-0078-9. URL: https://doi.org/10.1007/s40324-016-0078-9.

[73] Francisco Gancedo, Eduardo García-Juárez, Neel Patel, and Robert M. Strain. “Global Regularity for Gravity Unstable Muskat Bubbles”. In: *arXiv:1902.02318* (2019). URL: https://arxiv.org/abs/1902.02318.

[74] Francisco Gancedo, Eduardo García-Juárez, Neel Patel, and Robert M. Strain. “On the Muskat problem with viscosity jump: global in time results”. In: *Adv. Math.* 345 (2019), pp. 552–597. ISSN: 0001-8708. DOI: 10.1016/j.aim.2019.01.017. URL: https://doi.org/10.1016/j.aim.2019.01.017.

[75] Björn Gebhard and József J. Kolumbán. “Relaxation of the Boussinesq system and applications to the Rayleigh-Taylor instability”. In: *arXiv:2008.08853* (2020). URL: https://arxiv.org/abs/2008.08853.

[76] Björn Gebhard, József J. Kolumbán, and László Székelyhidi Jr. “A new approach to the Rayleigh-Taylor instability”. In: *arXiv:2002.08843* (2020).

[77] Nicola Gigli and Felix Otto. “Entropic Burgers’ equation via a minimizing movement scheme based on the Wasserstein metric”. In: *Calc. Var. Partial Differential Equations* 47.1-2 (2013), pp. 181–206. ISSN: 0944-2669. DOI: 10.1007/s00526-012-0515-2. URL: https://doi.org/10.1007/s00526-012-0515-2.

[78] Mikhael Gromov. *Partial differential relations*. Vol. 9. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1986, pp. x+363. ISBN: 3-540-12177-3. DOI: 10.1007/978-3-662-02267-2. URL: https://doi.org/10.1007/978-3-662-02267-2.

[79] Lauri Hitruhin and Sauli Lindberg. “Lamination Convex Hull of Stationary Incompressible Porous Media Equations”. In: *SIAM J. Math. Anal.* 53.1 (2021), pp. 491–508. ISSN: 0036-1410. DOI: 10.1137/20M134633X. URL: https://doi.org/10.1137/20M134633X.

[80] G. M. Homsy. “Viscous Fingering in Porous Media”. In: *Ann. Review Fluid Mech.* 19.1 (1987), pp. 271–311. DOI: 10.1146/annurev.fl.19.010187.001415. URL: https://doi.org/10.1146/annurev.fl.19.010187.001415.

[81] Philip Isett. “A proof of Onsager’s conjecture”. In: *Ann. of Math. (2)* 188.3 (2018), pp. 871–963. ISSN: 0003-486X. DOI: 10.4007/annals.2018.188.3.4. URL: https://doi.org/10.4007/annals.2018.188.3.4.

[82] Philip Isett and Vlad Vicol. “Hölder continuous solutions of active scalar equations”. In: *Ann. PDE* 1.1 (2015), Art. 2, 77. ISSN: 2524-5317. DOI: 10.1007/s40818-015-0002-0. URL: https://doi.org/10.1007/s40818-015-0002-0.

[83] Matt Jacobs, Inwon Kim, and Alpár R. Mészáros. "Weak Solutions to the Muskat Problem with Surface Tension Via Optimal Transport". In: *Arch. Ration. Mech. Anal.* 239.1 (2021), pp. 389–430. ISSN: 0003-9527. DOI: 10.1007/s00205-020-01579-3. URL: https://doi.org/10.1007/s00205-020-01579-3.

[84] Bernd Kirchheim. "Rigidity and geometry of microstructures". PhD thesis. Universtät Leipzig, 2003. URL: https://www.mis.mpg.de/de/publications/andere-reihen/ln/lecturenote-1603.html.

[85] Gereon Knott. "Oscillatory solutions to hyperbolic conservation laws and active scalar equations". PhD thesis. Universtät Leipzig, 2013.

[86] Robert Krasny. "Computation of vortex sheet roll-up in the Trefftz plane". In: *J. Fluid Mech.* 184 (1987), pp. 123–155. DOI: 10.1017/S0022112087002830. URL: https://doi.org/10.1017/S0022112087002830.

[87] Robert Krasny. "Desingularization of periodic vortex sheet roll-up". In: *J. Comput. Phys.* 65 (1986), pp. 292–313. DOI: 10.1016/0021-9991(86)90210-X. URL: https://doi.org/10.1016/0021-9991(86)90210-X.

[88] Nicolaas H. Kuiper. "On C^1-isometric imbeddings. I, II". In: *Nederl. Akad. Wetensch. Proc. Ser. A.* **58** = *Indag. Math.* 17 (1955), pp. 545–556, 683–689.

[89] Pijush Kundu, Ira Cohen, and David Dowling. *Fluid Mechanics*. Sixth. Elsevier/Academic Press, 2016, pp. xxiv+922. ISBN: 978-0-12-405935-1.

[90] Gilles Lebeau. "Régularité du problème de Kelvin-Helmholtz pour l'équation d'Euler 2d". In: *Séminaire: Équations aux Dérivées Partielles, 2000–2001*. Sémin. Équ. Dériv. Partielles. École Polytech., Palaiseau, 2001, Exp. No. II, 12.

[91] Pierre-Louis Lions. *Mathematical topics in fluid mechanics. Vol. 1*. Vol. 3. Oxford Lecture Series in Mathematics and its Applications. Incompressible models, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1996, pp. xiv+237. ISBN: 0-19-851487-5.

[92] Jian-Guo Liu and Zhou Ping Xin. "Convergence of vortex methods for weak solutions to the 2-D Euler equations with vortex sheet data". In: *Comm. Pure Appl. Math.* 48.6 (1995), pp. 611–628. ISSN: 0010-3640. DOI: 10.1002/cpa.3160480603. URL: https://doi.org/10.1002/cpa.3160480603.

[93] Milton C. Lopes Filho, Helena J. Nussenzveig Lopes, and Steven Schochet. "A criterion for the equivalence of the Birkhoff-Rott and Euler descriptions of vortex sheet evolution". In: *Trans. Amer. Math. Soc.* 359.9 (2007), pp. 4125–4142. ISSN: 0002-9947. DOI: 10.1090/S0002-9947-07-04309-7. URL: https://doi.org/10.1090/S0002-9947-07-04309-7.

[94] Milton C. Lopes Filho, Helena J. Nussenzveig Lopes, and Zhouping Xin. "Existence of vortex sheets with reflection symmetry in two space dimensions". In: *Arch. Ration. Mech. Anal.* 158.3 (2001), pp. 235–257. ISSN: 0003-9527. DOI: 10.1007/s002050100145. URL: https://doi.org/10.1007/s002050100145.

[95] Andrew J. Majda. "Remarks on weak solutions for vortex sheets with a distinguished sign". In: *Indiana Univ. Math. J.* 42.3 (1993), pp. 921–939. ISSN: 0022-2518. DOI: 10.1512/iumj.1993.42.42043. URL: https://doi.org/10.1512/iumj.1993.42.42043.

[96] Andrew J. Majda and Andrea L. Bertozzi. *Vorticity and incompressible flow.* Vol. 27. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2002, pp. xii+545. ISBN: 0-521-63057-6; 0-521-63948-4.

[97] O. Manickam and George M. Homsy. "Fingering instabilities in vertical miscible displacement flows in porous media". In: *J. Fluid. Mech.* 288 (1995), pp. 75–102. DOI: `10.1017/S0022112095001078`. URL: `https://doi.org/10.1017/S0022112095001078`.

[98] Simon Markfelder. "Convex Integration Applied to the Multi-Dimensional Compressible Euler Equations". PhD thesis. Julius-Maximilians-Universität Würzburg, 2021.

[99] Simon Markfelder and Christian Klingenberg. "The Riemann problem for the multidimensional isentropic system of gas dynamics is ill-posed if it contains a shock". In: *Arch. Ration. Mech. Anal.* 227.3 (2018), pp. 967–994. ISSN: 0003-9527. DOI: `10.1007/s00205-017-1179-z`. URL: `https://doi.org/10.1007/s00205-017-1179-z`.

[100] Bogdan-Vasile Matioc. "The Muskat problem in two dimensions: equivalence of formulations, well-posedness, and regularity results". In: *Anal. PDE* 12.2 (2019), pp. 281–332. ISSN: 2157-5045. DOI: `10.2140/apde.2019.12.281`. URL: `https://doi.org/10.2140/apde.2019.12.281`.

[101] Bogdan-Vasile Matioc. "Viscous displacement in porous media: the Muskat problem in 2D". In: *Trans. Amer. Math. Soc.* 370.10 (2018), pp. 7511–7556. ISSN: 0002-9947. DOI: `10.1090/tran/7287`. URL: `https://doi.org/10.1090/tran/7287`.

[102] Francisco Mengual. "H-principle for the 2D incompressible porous media equation with viscosity jump". In: *Anal. PDE* (To appear).

[103] Francisco Mengual and László Székelyhidi Jr. "Dissipative Euler flows for vortex sheet initial data without distinguished sign". In: *Comm. Pure Appl. Math.* (To appear).

[104] Stefano Modena and Gabriel Sattig. "Convex integration solutions to the transport equation with full dimensional concentration". In: *Ann. Inst. H. Poincaré Anal. Non Linéaire* 37.5 (2020), pp. 1075–1108. ISSN: 0294-1449. DOI: `10.1016/j.anihpc.2020.03.002`. URL: `https://doi.org/10.1016/j.anihpc.2020.03.002`.

[105] Stefano Modena and László Székelyhidi Jr. "Non-renormalized solutions to the continuity equation". In: *Calc. Var. Partial Differential Equations* 58.6 (2019), Paper No. 208, 30. ISSN: 0944-2669. DOI: `10.1007/s00526-019-1651-8`. URL: `https://doi.org/10.1007/s00526-019-1651-8`.

[106] Stefano Modena and László Székelyhidi Jr. "Non-uniqueness for the transport equation with Sobolev vector fields". In: *Ann. PDE* 4.2 (2018), Paper No. 18, 38. ISSN: 2524-5317. DOI: `10.1007/s40818-018-0056-x`. URL: `https://doi.org/10.1007/s40818-018-0056-x`.

[107] Derek W. Moore. "The spontaneous appearance of a singularity in the shape of an evolving vortex sheet". In: *Proc. Roy. Soc. London Ser. A* 365.1720 (1979), pp. 105–119. ISSN: 0962-8444. DOI: `10.1098/rspa.1979.0009`. URL: `https://doi.org/10.1098/rspa.1979.0009`.

[108] Stefan Müller and Vladimír Šverák. "Convex integration for Lipschitz mappings and counterexamples to regularity". In: *Ann. of Math. (2)* 157.3 (2003), pp. 715–742. ISSN: 0003-486X. DOI: `10.4007/annals.2003.157.715`. URL: `https://doi.org/10.4007/annals.2003.157.715`.

[109] Morris Muskat. *The Flow of Homogeneous Fluids Through Porous Media.* New York: McGraw-Hill, 1937. ISBN: 978-0934634168.

[110] Morris Muskat. "Two Fluid Systems in Porous Media. The Encroachment of Water into an Oil Sand". In: *Physics* 5 (1934), pp. 250–264. DOI: 10.1063/1.1745259. URL: https://doi.org/10.1063/1.1745259.

[111] John Nash. "C^1 isometric imbeddings". In: *Ann. of Math. (2)* 60 (1954), pp. 383–396. ISSN: 0003-486X. DOI: 10.2307/1969840. URL: https://doi.org/10.2307/1969840.

[112] Huy Q. Nguyen and Benoît Pausader. "A paradifferential approach for well-posedness of the Muskat problem". In: *Arch. Ration. Mech. Anal.* 237.1 (2020), pp. 35–100. ISSN: 0003-9527. DOI: 10.1007/s00205-020-01494-7. URL: https://doi.org/10.1007/s00205-020-01494-7.

[113] Florent Noisette and László Székelyhidi Jr. "Mixing solutions for the Muskat problem with variable speed". In: *J. Evol. Equ.* (2020). DOI: 10.1007/s00028-020-00655-1. URL: https://doi.org/10.1007/s00028-020-00655-1.

[114] Lars Onsager. "Statistical hydrodynamics". In: *Nuovo Cimento (9)* 6.Supplemento, 2 (Convegno Internazionale di Meccanica Statistica) (1949), pp. 279–287. ISSN: 0029-6341.

[115] Felix Otto. "Evolution of microstructure: an example". In: *Ergodic theory, analysis, and efficient simulation of dynamical systems.* Springer, Berlin, 2001, pp. 501–522.

[116] Felix Otto. "Evolution of microstructure in unstable porous media flow: a relaxational approach". In: *Comm. Pure Appl. Math.* 52.7 (1999), pp. 873–915. ISSN: 0010-3640. DOI: 10.1002/(SICI)1097-0312(199907)52:7<873::AID-CPA5>3.3.CO;2-K. URL: https://doi.org/10.1002/(SICI)1097-0312(199907)52:7<873::AID-CPA5>3.3.CO;2-K.

[117] John C. Oxtoby. *Measure and category.* Second. Vol. 2. Graduate Texts in Mathematics. A survey of the analogies between topological and measure spaces. Springer-Verlag, New York-Berlin, 1980, pp. x+106. ISBN: 0-387-90508-1.

[118] Bruce R. Pearson, Per-Åge Krogstad, and Willem van de Water. "Measurements of the turbulent energy dissipation rate". In: *Physics of Fluids* 14.3 (2002), pp. 1288–1290. DOI: 10.1063/1.1445422. URL: https://doi.org/10.1063/1.1445422.

[119] Philip G. Saffman and Geoffrey Taylor. "The penetration of a fluid into a porous medium or Hele-Shaw cell containing a more viscous liquid". In: *Proc. Roy. Soc. London Ser. A* 245 (1958), 312–329. (2 plates). ISSN: 0962-8444. DOI: 10.1098/rspa.1958.0085. URL: https://doi.org/10.1098/rspa.1958.0085.

[120] Steven Schochet. "The point-vortex method for periodic weak solutions of the 2-D Euler equations". In: *Comm. Pure Appl. Math.* 49.9 (1996), pp. 911–965. ISSN: 0010-3640. DOI: 10.1002/(SICI)1097-0312(199609)49:9<911::AID-CPA2>3.0.CO;2-A. URL: https://doi.org/10.1002/(SICI)1097-0312(199609)49:9<911::AID-CPA2>3.0.CO;2-A.

[121] Steven Schochet. "The weak vorticity formulation of the 2-D Euler equations and concentration cancellation". In: *Comm. Partial Differential Equations* 20.5-6 (1995), pp. 1077–1104. ISSN: 0360-5302. DOI: 10.1080/03605309508821124. URL: https://doi.org/10.1080/03605309508821124.

[122] Roman Shvydkoy. "Convex integration for a class of active scalar equations". In: *J. Amer. Math. Soc.* 24.4 (2011), pp. 1159–1174. ISSN: 0894-0347. DOI: 10.1090/S0894-0347-2011-00705-4. URL: https://doi.org/10.1090/S0894-0347-2011-00705-4.

[123] Michael Siegel, Russel E. Caflisch, and Sam Howison. "Global existence, singular solutions, and ill-posedness for the Muskat problem". In: *Comm. Pure Appl. Math.* 57.10 (2004), pp. 1374–1411. ISSN: 0010-3640. DOI: 10.1002/cpa.20040. URL: https://doi.org/10.1002/cpa.20040.

[124] Katepalli R. Sreenivasan. "An update on the energy dissipation rate in isotropic turbulence". In: *Physics of Fluids* 10.2 (1998), pp. 528–529. DOI: 10.1063/1.869575. URL: https://doi.org/10.1063/1.869575.

[125] Catherine Sulem, Pierre-Louis Sulem, Claude W. Bardos, and Uriel Frisch. "Finite time analyticity for the two- and three-dimensional Kelvin-Helmholtz instability". In: *Comm. Math. Phys.* 80.4 (1981), pp. 485–516. ISSN: 0010-3616. URL: http://projecteuclid.org/euclid.cmp/1103920098.

[126] László Székelyhidi Jr. "Relaxation of the incompressible porous media equation". In: *Ann. Sci. Éc. Norm. Supér. (4)* 45.3 (2012), pp. 491–509. ISSN: 0012-9593. DOI: 10.24033/asens.2171. URL: https://doi.org/10.24033/asens.2171.

[127] László Székelyhidi Jr. "Weak solutions to the incompressible Euler equations with vortex sheet initial data". In: *C. R. Math. Acad. Sci. Paris* 349.19-20 (2011), pp. 1063–1066. ISSN: 1631-073X. DOI: 10.1016/j.crma.2011.09.009. URL: https://doi.org/10.1016/j.crma.2011.09.009.

[128] László Székelyhidi Jr. and Emil Wiedemann. "Young measures generated by ideal incompressible fluid flows". In: *Arch. Ration. Mech. Anal.* 206.1 (2012), pp. 333–366. ISSN: 0003-9527. DOI: 10.1007/s00205-012-0540-5. URL: https://doi.org/10.1007/s00205-012-0540-5.

[129] Luc Tartar. "Compensated compactness and applications to partial differential equations". In: *Nonlinear analysis and mechanics: Heriot-Watt Symposium, Vol. IV.* Vol. 39. Res. Notes in Math. Pitman, Boston, Mass.-London, 1979, pp. 136–212.

[130] Luc Tartar. "Incomprensible fluid flow in a porous medium - Convergence of the homogenization processs". In: *Nonhomogeneous media and vibration theory.* Appendix. Springer-Verlag, Berlin-New York, 1980. ISBN: 3-540-10000-8.

[131] Gretar Tryggvason and Hassan Aref. "Numerical experiments on Hele-Shaw flow with sharp interface". In: *J. Fluid Mech.* 136 (1983), pp. 1–30. DOI: 10.1017/S0022112083002037. URL: https://doi.org/10.1017/S0022112083002037.

[132] Milton Van Dyke. *An album of fluid motion.* The Parabolic Press. Stanford University, Stanford, California, 1982.

[133] Italo Vecchi and Sijue Wu. "On L^1-vorticity for 2-D incompressible flow". In: *Manuscripta Math.* 78.4 (1993), pp. 403–412. ISSN: 0025-2611. DOI: 10.1007/BF02599322. URL: https://doi.org/10.1007/BF02599322.

[134] Robin A. Wooding and H. J. Morel-Seytoux. "Multiphase Fluid Flow Through Porous Media". In: *Ann. Review Fluid Mech.* 8.1 (1976), pp. 233–274. DOI: 10.1146/annurev.fl.08.010176.001313. URL: https://doi.org/10.1146/annurev.fl.08.010176.001313.

[135] Sijue Wu. “Mathematical analysis of vortex sheets”. In: *Comm. Pure Appl. Math.* 59.8 (2006), pp. 1065–1206. ISSN: 0010-3640. DOI: 10.1002/cpa.20110. URL: https://doi.org/10.1002/cpa.20110.

[136] Fahuai Yi. “Global classical solution of Muskat free boundary problem”. In: *J. Math. Anal. Appl.* 288.2 (2003), pp. 442–461. ISSN: 0022-247X. DOI: 10.1016/j.jmaa.2003.09.003. URL: https://doi.org/10.1016/j.jmaa.2003.09.003.